Topologically Twisted Yang-Mills Theory on K3 Surfaces

Master Thesis
Sietse Ringers
Supervisor: Prof. Dr. R. H. Dijkgraaf

UNIVERSITY OF AMSTERDAM
FACULTY OF SCIENCE

May 24, 2011

Contents

Introduction

In the last few decades, the relationship between mathematics and physics has changed. Historically, the advance of physics stimulated the development of a number of areas of mathematics, but the influence did not go much further than that. However, in the last thirty or forty years or so, it has started to happen more and more often that physical theories and ideas suddenly turn out to have uses in pure mathematics. A striking example of this is S. K. Donaldson's use of gauge theory to study four-dimensional differential geometry. Mysteriously, the four-dimensional case has turned out to be wildly complicated; both the lower and higher dimensional cases are much easier to deal with. On the other hand, much of the mathematics that one uses for gauge theories takes a particular simple form in the four-dimensional case. Indeed, much of the literature on this subject specifically targets four-dimensional spaces.

Physically, there is of course also plenty of reason to study gauge theories. It was first realized by C. N. Yang and R. Mills that much of particle physics could be described by it, and since then the subject has flourished. The simplest example of a gauge theory is electromagnetism. This particular example also shows a fascinating symmetry: the only thing that keeps Maxwell's equations, which completely describe the theory, from being completely symmetric in electricity and magnetism is the absence of magnetic monopoles. If those were to exist, one could switch electricity and magnetism and obtain the exact same theory. A natural question is then whether such a symmetry exists in more complicated examples; for example, this has been extensively researched for supersymmetric Yang-Mills theories, with largely affirmative results. In this context, this symmetry is called S-duality. Mathematically, in order for a gauge theory to be S-dual, the relevant quantities have to be so-called modular forms, which is a highly non-trivial property. This connects gauge theories to fascinating mathematical areas such as number theory and the Langlands program.

However, describing Yang-Mills theories in full detail is complicated. To address this issue, a technique exist called topological twisting: by subtly changing the theory one obtains a new theory, which is easier to deal with. Mathematically, the twisted theory is also interesting: the modification of the theory is done in such a way that the relevant physical quantities, such as the partition function, become topological invariants: entities that encode topological information on the background space. Physically, the topologically twisted theory is still interesting, because even though it no longer describes actual reality, it still contains much of the relevant features that a physical theory should have, and in a few particular cases, including ours, it even coincides with the original theory. It has, in effect, become a highly useful and interesting toy model. The twisting does not affect the S-duality properties of the theory, so these twisted theories can be used to find out to what degree Yang-Mills theories are S-dual. This was first done by C. Vafa and E. Witten [1].

Summarizing, right on the boundary of mathematics and physics there is a fascinating and highly fruitful connection between four-dimensional geometry and gauge theory.[1] This thesis positions itself precisely on this boundary by studying two particular examples of such a topologically twisted gauge theory, and whether they are S-dual to each other. We will do this by calculating the partition function of both theories, and try to determine if they are S-dual.

[1] For a more comprehensive discussion on this boundary, and more fascinating examples, see for example [2].

Outline

In the case of electromagnetism, the object of study is the electromagnetic four-potential $A_\mu = (\phi, \boldsymbol{A})$, called the *gauge potential*. The electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are usually combined into an antisymmetric two-tensor $F_{\mu\nu}$, called the *field strength*. $F_{\mu\nu}$ can be derived from A_μ, and it determines how an electrically charged particle moves in the electromagnetic field. Concretely, in units where $c = 0$, Maxwell's equations are,

$$\begin{aligned} \nabla \cdot \boldsymbol{E} &= 4\pi\rho_e & -\nabla \times \boldsymbol{E} &= \frac{\partial \boldsymbol{B}}{\partial t} \\ \nabla \cdot \boldsymbol{B} &= 0 & \nabla \times \boldsymbol{B} &= 4\pi \boldsymbol{J}_e + \frac{\partial \boldsymbol{E}}{\partial t}. \end{aligned}$$

Here, ρ_e is the electric charge density, and $\boldsymbol{J}_e$ the current density. Because there does not seem to be such a thing as magnetic monopoles, there are no magnetic charge and current densities ρ_m and $\boldsymbol{J}_m$. If there *would* be magnetic monopoles, however, then we would have

$$\begin{aligned} \nabla \cdot \boldsymbol{E} &= 4\pi\rho_e & -\nabla \times \boldsymbol{E} &= 4\pi \boldsymbol{J}_m + \frac{\partial \boldsymbol{B}}{\partial t} \\ \nabla \cdot \boldsymbol{B} &= 4\pi\rho_m & \nabla \times \boldsymbol{B} &= 4\pi \boldsymbol{J}_e + \frac{\partial \boldsymbol{E}}{\partial t}. \end{aligned}$$

Up to a minus sign (which disappears after using a Wick rotation to go over to a Euclidean theory), these equations are symmetric under exchanging electricity with magnetism. It was found by P. A. M. Dirac that in quantum theory, if there exists such a thing as magnetic monopoles having charge g_m, then this charge must be of the form (in natural units $\hbar = c = 1$) $g_m = \frac{1}{2}n/e$, where n is some natural number. The symmetry that switches electricity with magnetism is called *S-duality*. We see that if magnetic monopoles would exist, then Euclidean electromagnetism would indeed be S-dual, and moreover that electric charge would be quantized, which indeed seems to be the case. Concretely, any quantity that depends on the coupling constant has to be left invariant when one sends the coupling constant to its inverse. Mathematically, this is a highly nontrivial property, having to do with modular forms.

Electromagnetism is a gauge theory which has gauge group $U(1)$. In more general theories, having more complicated and non-abelian gauge groups, one still studies the gauge potential A, but now it takes its values in the Lie algebra of the gauge group, and due to topological obstructions it may only be specified locally. Mathematically, A has a geometrical interpretation as specifying the horizontal direction on fiber bundles; we shall treat this formalism in the first two chapters. Locally, A still induces a field strength F. The *Yang-Mills action* S_{YM}, which determines when a gauge field A corresponds to a possible physical configuration, is an integral over this field-strength F.

Now, there exists an operator $\star$ called the *Hodge dual*, which in the case of electromagnetism roughly sends $\boldsymbol{E}$ to $\boldsymbol{B}$ and vice versa. Mathematically, it acts on the space of two-forms, of which F is an element, and in F it switches the electric and magnetic components around. This implies that it squares to 1, which means that the space of two-forms splits into eigenspaces with eigenvalues $+1$ and -1. We call eigenvectors of this kind *(anti-)self-dual*, and we say that the gauge potential A is an *instanton*. They are called instantons because it turns out that they describe a phenomenon that is not only localized in space, but also in time.

In Chapter 4 we shall see that if F is an eigenvector of the Hodge dual $\star$, i.e. an instanton, then it is an absolute minimum of the Yang-Mills action S_{YM}. Since the Euclidean partition function Z is of the form $Z = \int DA\, e^{-S[A]/\hbar}$, these instantons give a large contribution to the partition function. Therefore, it plays an important role in the quantum theory. Moreover, the (anti-)self-duality equation $F = \pm\star F$ is more easily solved than the demand that F minimizes the action, because it is a differential equation of first order, while the demand that F minimizes the Yang-Mills action S_{YM} yields a differential equation of second order. Mathematically, self-dual field strengths are also important, because it turns out that when a field strength is self-dual, then the action S_{YM} is an integer, called the *instanton number*, which is an invariant of the underlying bundle. Furthermore, when the gauge group is $\mathrm{SU}(2)$ (which will be the case in a large part of this thesis) it determines the bundle completely. That is to say, any integer number n uniquely determines the topological and differential structure of the space on which A and F live. This is a fascinating result: given any self-dual connection, one can discover through it exactly the structure of the underlying space.

In the case of electromagnetism, there is the concept of *gauge freedom*: if A_μ is a gauge potential, and f is a function depending on spacetime, then one can add the derivative of f to A: $A'_\mu = A_\mu + \partial_\mu f$. Physically, these determine the same configuration; there is no measurable difference between them (classically, at least). This extends to the case of more complicated gauge groups. The *gauge group* $\mathscr{G}$ is a group that acts on the gauge potential A, giving a new gauge potential A'. If two gauge potentials can be related to each other by a gauge transformation, then they determine the same physical configuration. In other words, the theory contains intrinsic degrees of freedom, which one must divide out by considering not gauge potentials themselves, but the orbits of gauge potentials under the gauge group $\mathscr{G}$.

A natural space to study, then, is the space of instantons, modulo these gauge transformations. This space is called the *instanton moduli space*. It is important for physicists because it contains information on the minima of the action, and to mathematicians because of the role that instantons play in studying the underlying space. We shall explore these spaces in Chapter 5; they are important to us, because it turns out that the influence that instantons have on the partition function can be described in terms of topological constants of these moduli spaces.

After having explored all of this, in Chapter 6 we shall do the actual calculations. First we study in more detail what S-duality means in our particular case. It turns out that not only does the coupling constant g get sent to its inverse $-1/g$, but the gauge group $\mathrm{SU}(2)$ is also replaced by $\mathrm{SO}(3)$. Therefore, we shall have to calculate the partition functions for both of these theories. However, calculating the full partition function is hard. To deal with this, there exists a technique called *topological twisting*, that modifies the theory to make it easier to handle, while at the same time increasing its mathematical interest. In essence, one adds a number of degrees of freedom to the theory, in such a way that the partition function Z becomes a topological invariant. Moreover, instead of just being the dominant contributions, it turns out that in the twisted theory the instantons are the *only* contribution to the partition function. That is to say, if one writes Z as a saddle approximation around the instantons, then this approximation is exact, so Z reduces to a sum over the instantons, which is clearly much easier to compute. Concretely, it will be of the form

$$Z = \sum_n q^n a_n, \qquad q = e^{2i\pi\tau},$$

where τ is the coupling constant, and where a_n are constants determined by the moduli space of instantons having instanton number n.

Generally, an oriented smooth 4-manifold has structure group and holonomy $K = \mathrm{SO}(4)$, which is locally isomorphic to $\mathrm{SU}(2) \otimes \mathrm{SU}(2)$. The $\mathcal{N} = 4$ supersymmetry adds to this a group $\mathrm{SU}(4)$, so the full symmetry group is of the form $\mathrm{SU}(2) \otimes \mathrm{SU}(2) \otimes \mathrm{SU}(4)$. Topological twisting essentially modifies the two factors on the right, leaving the left $\mathrm{SU}(2)$ untouched. Therefore, it makes sense to take as base space one of which the structure group fits entirely in the left factor of $\mathrm{SO}(4) \cong \mathrm{SU}(2) \otimes \mathrm{SU}(2)$; that is, a Calabi-Yau manifold. In complex dimension 2 (i.e. real dimension 4), the simplest examples of these are *K3 surfaces*; they are the only Calabi-Yau manifolds that are compact and simply connected. Therefore, we shall take this as our base space.

We shall see that the $\mathrm{SO}(3)$ partition function can be written as a sum over 2^{22} terms, all of which satisfy the expected S-duality conditions. We will be able to compute almost all of these, but one will give complications. This term consists of contributions from moduli spaces which are not smooth, for which we find no convenient description. Therefore, in order to be able to calculate this last term (and through it, the $SU(2)$ partition function), we will assume S-duality. This determines the term completely, finishing the calculation. Since the publication of [1], the S-duality conjecture has been shown to hold for this particular case (and in fact in other cases too).

Acknowledgements

First, I would like to express my extensive gratitude to my supervisor, R. Dijkgraaf, for his time and guidance, and for sharing his insight in these theories. His wisdom and enthusiasm were very inspiring, at a time when I needed inspiring the most.

Second, I would like to thank G. van der Geer, professor of algebra of the University at Amsterdam, for his clear explanation of K3 surfaces.

Lastly, but certainly not least, much thanks go to my fellow student and friend A. J. Lindenhovius, for the excellent collaboration that we enjoyed in the first few months of this project, his help and support, and the huge amount of attentive suggestions and corrections.

Prerequisites

The reader should be familiar with Riemannian and pseudo-Riemannian geometry, and Lie groups, Lie algebras, and their relationships. Some familiarity with bundles would help but is not strictly necessary, as some bundle theory will be treated in the first two chapters. Physical prerequisites are quantum mechanics, quantum field theory, and electrodynamics.

Chapter 1

Fiber bundles

In this chapter, we introduce various kinds of fiber bundles, which are the spaces on which the rest of the formalism of this thesis is based.

A fiber bundle is a generalization of product spaces, in the sense that it locally looks like a product space of two other spaces (called the base space and the fiber). Globally, however, it generally has a more complicated structure. Different kinds of bundles are obtained by considering different kinds of fibers; however, the two that are most important to us, principal bundles and vector bundles, will turn out to be intimately connected.

Then we will discuss gauge transformations. These are morphisms from the fiber bundle to itself that preserve their structure in a certain way. They will turn out to be related to connections, introduced in the next chapter, and they play an important role in the physics that one describes using this formalism.

For a more comprehensive introduction on fiber bundles, see for example [3, 4], and [5, chapter 9].

1.1 General bundles

Recall that a fiber bundle E consists of the following data:[1]

- Four smooth manifolds: E, called the *total space*; M, the *base space*; F, the *fiber*; and G, the *structure group*. G is assumed to be a matrix Lie group.
- An effective left group action of G on F.
- A smooth surjection $\pi : E \to M$, called the *projection*, such that $\pi^{-1}(x) \cong F$ for all $x \in M$.
- An open covering $\{U_i\}_i$ of M, and a set of diffeomorphisms $\psi_i : U_i \times F \to \pi^{-1}(U_i)$, called *local trivializations*.
- Let $x \in U_i \cap U_j$ and $f \in F$. Denoting $\psi_{i,x} : F \to E, f \mapsto \psi_{i,x}(f) = \psi_i(x, f)$, the maps $\psi_{i,x}^{-1}\psi_{j,x} : F \to F$ are required to be diffeomorphisms, coinciding with the action of some unique element $g_{ij}(x) \in G$ on F. The smooth maps $g_{ij} : M \to G$ are called the *transition functions*.

Thus, on overlaps we have

$$\psi_j(x, f_j) = \psi_i\left(x, \psi_{i,x}^{-1}\psi_{j,x}(f_j)\right) = \psi_i(x, g_{ij}(x)f_j). \tag{1.1}$$

[1]Strictly speaking, this definition should be independent of the open covering $\{U_i\}_i$. As it is, this definition is not an actual fiber bundle, but a coordinate bundle. If two coordinate bundles having respectively $\{\psi_i\}_i$ and $\{\psi_i'\}_i$ as trivializations such that $\tilde{g}_{ij}(x) = \psi'^{-1}_{i,x} \circ \psi_{j,x}$ coincides with the action of some element of g, and the functions $\tilde{g}_{ij}$ are smooth, then we say that they are equivalent. A fiber bundle is then defined as the equivalence class of a coordinate bundle. In practice, however, we will always employ a certain open covering, and neglect to make this distinction. For a more careful treatment of this distinction, see [3].

In other words, if f_i is the F-'coordinate' associated to the trivialization ψ_i, then $f_i = g_{ij}(x) f_j$.

If P is diffeomorphic to $M \times F$, or equivalently if there exists a trivialization which is global, then P is said to be *trivial*.

Let (ψ_i, U_i) be a local trivialization. Recall that a *local section* is a smooth map $s : U_i \to E$ such that $\pi \circ s = \mathrm{id}$, or alternatively $s(x) \in \pi^{-1}(x)$ for all x. A local section always exists[2]; for example, take $x \mapsto \psi_i(x, f)$ for some constant element $f \in F$.

Note that π is a submersion, and that any section is an immersion. To see this, take a local trivialization (ψ_i, U_i) and a section $s : U_i \to \pi^{-1}(U_i)$. Then s is right-inverse of π which is smooth, so

$$\mathrm{id}_{T_x M} = (\mathrm{id}_M)_* = (\pi \circ s)_* = \pi_* \circ s_*,$$

so s_* is a smooth right-inverse of π_*. Thus s_* is an injection and π_* is a surjection. In the case of π, this results immediately extends to the entire bundle. It follows that $\dim \operatorname{im} \pi_* = \dim M$, so that $\dim \ker \pi_* = \dim G$.

Proposition 1.1 *The transition functions satisfy for any i, j, k and $x \in M$*

$$\begin{aligned} g_{ii}(x) &= e \in G, \\ g_{ij}(x) &= g_{ji}(x)^{-1}, \\ g_{ij}(x) g_{jk}(x) &= g_{ik}(x). \end{aligned} \tag{1.2}$$

The only data we need to know of a fiber bundle is the base space M and an open covering $\{U_i\}$ on it, the transition functions (which have to satisfy the identities in the above proposition), the fiber and the structure group. If we have this data, then the fiber bundle is

$$E = \coprod_i U_i \times F / \sim, \tag{1.3}$$

where $(x, f) \sim (x, g_{ij}(x) f)$ on overlaps $U_i \cap U_j$. The projection then is just $[(x, f)] \mapsto x$, and the local trivializations $\psi_i : (x, f) \mapsto [(x, f)]$.

Definition 1.2 Let $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} M'$ be two fiber bundles. A *bundle map* is a smooth map $F : E \to E'$ such that it maps each fiber $\pi^{-1}(x)$ onto $\pi'^{-1}(x')$ for some x'. In this case, F induces a map $f : M \to M'$ such that $f(x) = x'$, and $\pi' \circ F = f \circ \pi$.

Two bundles over a base space M are said to be *equivalent* if there exists a bundle map F which is a diffeomorphism, and such that the map from M to M that it induces is the identity.

Definition 1.3 Let $E \xrightarrow{\pi} M$ be a fiber bundle with fiber F, and let $f : N \to M$ be a smooth map. We call $f^* E = \{(x, p) \in N \times E \mid f(x) = \pi(p)\}$ the *pullback bundle* of E by f. As projection we take the projection on the first component, $\pi(x, p) = x$.

Let $\{U_i\}_i$ be local trivializations of M. Then $\{f^{-1}(U_i)\}$ defines an open covering of N such that $f^* E$ is locally trivial. Let $q = (x, p)$ be such that $\pi(p) = f(x) \in U_i$, and find a $f_i \in F$ such that $\psi_i^{-1}(p) = (\pi(p), f_i) = (f(x), f_i)$. Now define a trivialization on N by $\phi_i^{-1}(q) = (x, f_i)$. This is then a local trivialization on $f^{-1}(U_i)$. Lastly, for the transition functions on $f^* E$ we then have

$$\phi_{i,x}^{-1} \phi_{j,x}(f_i) = \psi_{i,f(x)}^{-1} \psi_{j,f(x)}(f_i) = g_{ij}(f(x)) f_i,$$

so the transition functions on $f^* E$ are just $g_{ij} \circ f$. Thus, $f^* E$ becomes a fiber bundle over N with fiber F and transition functions $g_{ij} \circ f$.

[2] A smooth global section, i.e. a section with all of M as its domain, does not necessarily exist. For example, we shall see below that in certain cases the existence of a global section implies that the bundle is trivial.

Theorem 1.4 *Let $E \xrightarrow{\pi} M$ be a fiber bundle, and let f and g be homotopic maps from N to M. Then f^*E and g^*E are equivalent bundles over N.*

For a proof, see [3].

Corollary 1.5 *If $E \xrightarrow{\pi} M$ is a fiber bundle with fiber F and M is contractible to a point, then E is trivial.*

Proof Since M is contractible to a point, there is a homotopy $F : I \times M \to M$ such that $F(0, x) = x_0$ for some fixed x_0, and $F(1, x) = x$. Write $F_t(x) = F(t, x)$ and consider the bundle F_0^*E. The transition functions of this bundle are $g_{ij} \circ F_0$, so they are all constant. By the first identity in Proposition 1.1, they have to be the identity element $e \in G$. This implies that on overlaps, the trivializations agree, so that they can be glued together to form a single global trivialization. In other words, F_0^*E is trivial, $F_0^*E = M \times F$. On the other hand, $F_1^*E = E$, so E is equivalent with $M \times F$ by the theorem above, and therefore trivial. □

In particular, note that any bundle over $\mathbb{R}^4$ (or $\mathbb{R}^{3,1}$) is trivial.

Definition 1.6 Let $E \xrightarrow{\pi} M$ be a fiber bundle with fiber F and structure group G. Then E admits a *reduction of its structure group* to H, where H is a subgroup of G, if E is equivalent to a bundle with structure group H. Equivalently, E admits such a reduction of G to H if there is a choice of local trivializations covering E such that the transition functions all take values in H.

For example, as we have seen in the proof of the above theorem, E is trivial if and only if the structure group is reducible to the trivial group. When E is a vector bundle, then there always exists a Riemannian metric on E; this is equivalent to the structure group always being reducible from $\mathrm{GL}(n)$ to $\mathrm{O}(n)$.

1.2 Vector bundles

A *vector bundle* is a (real or complex) fiber bundle, usually denoted E, in which the fiber is a vector space such that for every trivialization (U, ψ) and every $x \in M$, the diffeomorphism $F \to \pi^{-1}(x) : v \mapsto \psi(x, v)$ is linear (over $\mathbb{R}$ or $\mathbb{C}$). The dimension of the fiber is called the *rank* of the vector bundle. Examples of these are the tangent and cotangent space, and the space of p-forms on a manifold.

When the following additional requirements are met:

- M is a complex manifold (see Appendix A),
- π is holomorphic,
- the fiber is a complex vector space,
- the trivializations are biholomorphic,

then the bundle is said to be *holomorphic*.

Given finite-dimensional vector spaces F, F', we know how to form the following vector spaces: F^*, $F \otimes F'$, $F \oplus F'$, $\Lambda^k F$, and so on. Now, suppose that F and F' are the fibers of two vector bundles E and E', and denote the transition maps by g_{ij} and g'_{ij}. Thus the structure group G acts on F and F', and there exist natural representations of G on F^*, $F \otimes F'$, and so on. Applying these constructions fiberwise, we can construct the following fiber bundles:

- E^*, having fiber F^* on which G acts by $\rho(g^{-1})$ where ρ is the representation of G on F. Its transition functions are $(g_{ij}^\top)^{-1}$.
- $E \oplus E'$, having transition functions $g_{ij}(x) \oplus g_{ij'}(x)$ (considering the image of the transition functions as matrices).
- $E \otimes E'$, having transition functions $g_{ij}(x) \otimes g_{ij'}(x)$.
- $\bigwedge^k E$, having transition functions $\bigwedge^k g_{ij}(x)$.
- If r is the rank of the bundle, then $\bigwedge^r E$ has rank 1; i.e. it is a *line bundle*. This bundle is called the *determinant bundle*.

Let M be complex. The space of holomorphic n-forms on M, i.e. the determinant bundle of the holomorphic cotangent bundle, is called the *canonical bundle* of M, and denoted by K_M. It will become important to us when we start dealing with K3 surfaces in chapter 3.

Let f and g be vector bundle homomorphisms, i.e. bundle homomorphisms that are fiber-wise linear. If we have a sequence

$$0 \to E_1 \xrightarrow{f} E \xrightarrow{g} E_2 \to 0 \tag{1.4}$$

such that $\ker(g) = \mathrm{im}(f)$, then the sequence is called a *short exact sequence*. In this case there is a canonical isomorphism

$$\det(E) = \det(E_1) \otimes \det(E_2). \tag{1.5}$$

Now consider holomorphic line bundles. If we have two of these, say L and L', then their tensor product $L \otimes L'$ is still a line bundle. We see immediately that $L \otimes L^*$ is the trivial bundle $M \times \mathbb{C}$. In fact, the set of isomorphism classes of line bundles becomes an abelian group, with the tensor product as multiplication, the dual of a bundle as its inverse, and the trivial bundle as identity. This group is called the *Picard group* $\mathrm{Pic}(M)$.

1.3 Principal bundles

A *principal bundle* is a fiber bundle in which the fiber is the structure group. They are denoted by $P(M, G)$. In this case, the action of G on the fiber defines a right action on P. Let $p \in P$, and $p = \psi_i(x, g_i)$ for some trivialization ψ_i and $x, g_i \in M, G$. Then the right action of G on P is defined as

$$pg = \psi_i(x, g_i g). \tag{1.6}$$

This definition is independent of the trivialization. Indeed, suppose that ψ_j is another trivialization. Then

$$pg = \psi_i(x, g_i g) = \psi_i(x, g_{ij}(x) g_j g) = \psi_j(x, g_j g).$$

Proposition 1.7 *Write a local trivialization as $\psi_i^{-1}(p) = (\pi(p), g_i(p))$. Then*
- *The right action preserves fibers, $\pi(pg) = \pi(p)$;*
- *the function g_i is equivariant, $g_i(pg) = g_i(p)g$;*
- *The right action is transitive and free on fibers.*

Proof For the first item, $\pi(pg) = \pi(\psi_i(x, g_i(p)g)) = x = \pi(p)$.

For the second item, on the one hand $pg = \psi_i(\pi(pg), g_i(pg)) = \psi_i(\pi(p), g_i(pg))$, while on the other hand

$$pg = \psi_i(\pi(p), g_i(p))g = \psi_i(\pi(p), g_i(p)g)$$

so $g_i(pg) = g_i(p)g$.

Lastly, under the diffeomorphism $\pi^{-1}(p) \cong G$ the right action is just right multiplication which is transitive and free. □

Theorem 1.8 *Let G be a compact Lie group acting smoothly, freely and properly on a smooth manifold P. Then the orbit space $M := P/G$ is a topological manifold of dimension $\dim P - \dim G$, which has a smooth structure uniquely determined by the demand that the quotient map $\pi : P \to P/G$ be a submersion.*

The proof of this statement is rather long and technical, and may be found in [6, p. 218].

Corollary 1.9 *If $P(M, G)$ is a principal bundle with G compact, then $M \approx P/G$ and $\dim P = \dim M + \dim G$.*

Definition 1.10 Let $\psi : U \times G \to \pi^{-1}(U)$ be a trivialization. Then the *canonical section* associated to ψ is defined by $s(x) := \psi(x,e)$. Note that $\psi(x,g) = \psi(x,eg) = \psi(x,e)g = s(x)g$. Conversely, let s be any section over U. Since the right action is transitive and free, any $p \in \pi^{-1}(x)$ for $x \in U$ can be written as $s(x)g$ for some g. Therefore, any section also defines a trivialization, which we call the canonical trivialization associated to s.

Note that a global section would induce a global canonical trivialization, so we have the following.

Theorem 1.11 *A principal bundle is trivial if and only if it admits a global section.*

Let $E \to M$ be a real vector bundle of rank k. Then there is an important associated principal bundle, which we now discuss. Recall that a frame at a point $x \in M$ is an ordered basis for the vector space E_x. Denote the set of all frames at x by F_x.

Definition 1.12 The *frame bundle* of E, denoted by FE is defined as $FE = \coprod_{x\in M} F_x$.

If $e = (e_1, \ldots, e_k)$ is a frame above x, then there is an obvious projection map π defined by $\pi(e) = x$. Any other frame, i.e. any other element of $\pi^{-1}(x)$ may be written as (using matrix multiplication) ge, where $g \in \mathrm{GL}(k,\mathbb{R})$. Thus the fiber is $\mathrm{GL}(k,\mathbb{R})$. Now take a trivialization of E over $U_i \subset M$. This is equivalent to taking a frame $e_i = (e_{i1}, \ldots, e_{ik})$ of E over U_i. We define a trivialization $\psi_i : U_i \times \mathrm{GL}(n,\mathbb{R}) \to \pi^{-1}(U_i)$ for FE by $\psi_i(x,g) := g^{-1}e_i(x)$. This makes FE into a principal bundle.

Now take another trivialization $U_j \subset M$ overlapping with U_i. By definition, then, the associated frame e_j may be expressed as $e_j(x) = g_{ji}(x)e_i(x)$, where g_{ij} are the transition functions of E. We find

$$\psi_j(x,g) = g^{-1}e_j(x) = g^{-1}g_{ji}(x)e_i(x) = (g_{ij}(x)g)^{-1}e_i(x) = \psi_i(x, g_{ij}(x)g), \tag{1.7}$$

that is, FE has the same transition functions as E.

When taking a trivialization of E and using the Gram-Schmidt procedure pointwise on it, one obtains an orthonormal frame. One may then take as fiber $\mathrm{O}(n)$ instead of $\mathrm{GL}(n,\mathbb{R})$, since any other orthonormal frame may be expressed as an element of $\mathrm{O}(n)$ times the original one. This shows that the structure group is reducible to $\mathrm{O}(n)$. If E is orientable, then it may further be reduced to $\mathrm{SO}(n)$. This group is not simply connected. However, there exists a double cover of $\mathrm{SO}(n)$ called $\mathrm{Spin}(n)$, which is.

Definition 1.13 Let H be a Lie group, and let $f : H \to G$ be a surjective covering homomorphism, such that $\ker f \subset Z(H)$. We say that $P(M,G)$ has a *lift* to a principal bundle $Q(M,H)$ if there exists a bundle map $\hat{f} : Q \to P$ such that $\hat{f}(pg) = \hat{f}(p)f(g)$ for $p \in Q, g \in H$.

Definition 1.14 Suppose a vector bundle E is Riemannian and orientable, so that the structure group reduces to $\mathrm{SO}(k)$. If FE has a lift to a $\mathrm{Spin}(k)$-bundle, then we say that E is *spin*. When $E = TM$ for some smooth manifold M, then we say that M is spin.

An immediate question is when a SO-bundle is spin. We shall answer this question later, in section 2.5.6.

1.4 Associated bundles

Let $P(M,G)$ be a principal bundle, and suppose we have some manifold F, and a faithful representation $\rho : G \to \mathrm{Aut}(F)$. Then we can define an action of $g \in G$ on $P \times F$ by

$$(p,f)g = (pg, \rho(g^{-1})f). \tag{1.8}$$

Let us denote an element of the orbit of this action by $[p,f] := [(p,f)]$.

Definition 1.15 The *associated fiber bundle* $E_\rho = P \times_\rho F$ is a fiber bundle over M with fiber F, and defined to be

$$E_\rho = P \times_\rho F := (P \times F)/G, \tag{1.9}$$

in which $[p, f] = [pg, \rho(g^{-1})f]$. The projection is $\pi_E([p, f]) = \pi(p)$, and the local trivializations are $\psi_{E,i}(x, f) = [\psi_i(x, e), f]$.

This projection map is well defined. If $(pg, \rho(g^{-1})f)$ is another representative of $[p, f]$, then $\pi_E([pg, \rho(g^{-1})f]) = \pi(pg) = \pi(p) = \pi_E([p, f])$. As for the local trivializations, note that the F-'coordinate' of some point $[p, f] = [\psi_i(x, g_i), f]$ under the trivialization $\psi_{E,i}$ is $f_i = \rho(g_i)f$.

The transition functions of this bundle are just $\rho \circ g_{ij}$. Indeed, suppose $p = \psi_i(x, g_i)$. Then the F-coordinate of $[p, f]$ under $\psi_{E,i}$ is $f_i = \rho(g_i)f$ for any i, so

$$\begin{aligned}[p, f] &= \psi_{E,i}(x, f_i) = \psi_{E,i}(x, \rho(g_i)f) = \psi_{E,i}(x, \rho(g_{ij}(x)g_j)f) \\ &= \psi_{E,i}(x, \rho(g_{ij}(x))\rho(g_j)f) = \psi_{E,i}(x, \rho(g_{ij}(x))f_j).\end{aligned}$$

Thus, taking the associated fiber bundle $P \times_\rho F$ amounts to changing the fiber from G to F, while keeping the transition functions. In other words,

$$P \times_\rho F = \coprod_i U_i \times F / \sim \tag{1.10}$$

where $(x, f) \sim (x, \rho(g_{ij}(x))f)$ on overlaps. Furthermore, a section $s : M \to P \times_\rho F$ can be described by a family of functions $s_i : U_i \to F$ satisfying

$$s_i(x) = \rho(g_{ij}(x))s_j(x) \tag{1.11}$$

on overlaps.

If the fiber F is a vector bundle, then we restrict ρ to be a vector space representation, i.e. $\rho(g)$ must be a linear automorphism. Generally, by definition of fiber bundles, there is a diffeomorphism $f : \pi^{-1}(x) \to F$ for each x; in fact, if ψ is a local trivialization then $\psi(x, f) \mapsto f$ is such a diffeomorphism. We can use this diffeomorphism to transfer the linear structure on F to $\pi^{-1}(x)$. In other words, we endow $\pi^{-1}(x)$ with the unique vector space structure that makes f a linear map. By construction, f is then not only a diffeomorphism, but also a vector space isomorphism. This is made more explicit in the following definition.

Definition 1.16 Write an arbitrary element of $\pi^{-1}(x)$ as $f(v)$ for some $v \in F$. Since f is bijective, this can always be done. Then we define scalar multiplication on $\pi^{-1}(x)$ by $af(v) := f(av)$ for scalar a. If $f(w)$ is another element of $\pi^1(x)$, then we define their linear combination by $f(w) + f(w) := f(v + w)$.

This makes the associated bundle into a vector bundle.

When we have a vector bundle, we can take its associated frame bundle, which is a principal bundle. On the other hand, if we have a principal bundle and some vector space, we can take the associated vector bundle. We have seen that both of these constructions leave the transition functions intact, which are in a sense the essence of a bundle; they encode how much the bundle is 'twisted'. Essentially, then, there is not much difference between the two, since we can switch back and forth between the two without losing any critical information. Whichever one we take is a matter of convenience for the particular subject at hand.

Two particularly important examples of associated bundles which we will encounter later on, are $\operatorname{ad} P := P \times_{\operatorname{ad}} \mathfrak{g}$ and $\operatorname{Ad} P := P \times_{\operatorname{Ad}} G$. Here, $\operatorname{Ad}(g) : G \to G$ is the map defined by $\operatorname{Ad}(g)(h) = ghg^{-1}$, while $\operatorname{ad}(g) : \mathfrak{g} \to \mathfrak{g}$ is its derivative at $e \in G$. For matrix groups, $\operatorname{ad}(g)(X) = gXg^{-1}$.

1.5 Gauge transformations

Definition 1.17 A *gauge transformation* of a principal bundle $P(M, G)$ is a diffeomorphism $\Phi : P \to P$ such that it preserves fibers and is equivariant; respectively,

$$\pi(\Phi(p)) = \pi(p) \quad \text{and} \quad \Phi(pg) = \Phi(p)g. \tag{1.12}$$

We denote the group of all gauge transformations by $\mathscr{G}$ and call it the *gauge group*. The group operation is composition. Each Φ has an inverse since Φ is a diffeomorphism, and it is easy to see that Φ^{-1} is also equivariant and fiber-preserving.

Theorem 1.18 *The gauge group is equal to*

$$\mathscr{G} = C^\infty_{\mathrm{Ad}}(P, G) := \{\varphi \in C^\infty(P, G) \mid \varphi(pg) = g^{-1}\varphi(p)g \text{ for all } g \in G\} \tag{1.13}$$

and

$$\mathscr{G} = \Gamma^\infty(M, \mathrm{Ad}\, P), \tag{1.14}$$

where Γ^∞ denotes the space of smooth sections.

Proof In this proof, we write the local trivialization as $\psi_i^{-1}(p) = (\pi(p), g_i(p))$.

We will first show the first equality. For each p, $\Phi(p)$ lies in the same fiber as p. Then the transitiveness of the group action on the fiber implies that for each p there is a $\varphi(p)$ such that $\Phi(p) = p\varphi(p)$. Indeed, $\varphi(p) = g_i(p)^{-1}g_i(\Phi(p))$ satisfies this requirement. Moreover, since g_i and the group multiplication on G is smooth, Φ being smooth is equivalent with φ being smooth.

We must have $(pg)\varphi(pg) = \Phi(pg) = \Phi(p)g = p\varphi(p)g$, which implies $\varphi(pg) = g^{-1}\varphi(p)g$. Now let φ be a map $\varphi : P \to G$ satisfying this condition. If we then define $\Phi(p) = p\varphi(p)$, then Φ preserves fibers because the right action does, and

$$\Phi(pg) = pg\varphi(pg) = pgg^{-1}\varphi(p)g = p\varphi(p)g = \Phi(p)g,$$

so $\Phi \in \mathscr{G}$. Thus this correspondence is a bijection. Moreover, let Φ_1, Φ_2 be two gauge transformations and φ_1, φ_2 their corresponding elements of $C^\infty_{\mathrm{Ad}}(P, G)$. Then

$$(\Phi_1 \circ \Phi_2)(p) = \Phi_1(\Phi_2(p)) = \Phi_1(p\varphi_2(p)) = \Phi_1(p)\varphi_2(p) = p\varphi_1(p)\varphi_2(p).$$

Thus the correspondence is also a homomorphism.

As to the second equality, let $\{U_i, \psi_i\}_i$ be local trivializations, and Φ a gauge transformation. Since Φ maps fibers to itself, it restricts to a gauge transformation on the trivial bundle $\pi^{-1}U_i$ over U_i. Now for each overlap we define a function $\phi_i : \pi^{-1}U_i \to G$ by

$$\phi_i(p) := g_i(p)\varphi(p)g_i(p)^{-1} = g_i(\Phi(p))g_i(p)^{-1},$$

where φ is defined as in the first equality of this theorem, so that

$$\psi_i(\pi(p), \phi_i(p)g_i(p)) = \psi_i(\pi(p), g_i(p)\varphi(p)) = \psi_i(\pi(p), g_i(p))\varphi(p) = \Phi(p).$$

These then satisfy

$$\phi_i(pg) = g_i(p)gg^{-1}\varphi(p)gg^{-1}g_i(p)^{-1} = g_i(p)\varphi(p)g_i(p)^{-1} = \phi_i(p).$$

Since the action of G is simply transitive on the fiber, this implies that this function depends only on the base point $\pi(p)$, so it descends to a function $\phi_i : U_i \to G$. Thus, locally, a gauge transformation is equivalent to a map $U_i \to G$. Globally, the gauge transformation has to satisfy on overlaps $U_i \cap U_j$, taking any $p \in \pi^{-1}(x)$,

$$\phi_i(x) = g_i(p)\varphi(p)g_i(p)^{-1} = g_{ij}(x)g_j(p)\varphi(p)g_j(p)^{-1}g_{ij}(x)^{-1}$$

$$= g_{ij}(x)\phi_j(x)g_{ij}^{-1}(x), \tag{1.15}$$

showing that $\{\phi_i\}_i \in C^\infty(M, \operatorname{Ad} P)$ (see the remarks at the end of section 1.4). From their definition, it is clear that a set of $\{\phi_i\}_i$ defined on all of M satisfying this condition completely determines a φ and thus an element of $\mathscr{G}$. This correspondence is also a homomorphism:

$$(\Phi_1 \circ \Phi_2)(p) = \Phi_1(\Phi_2(p)) = \Phi_1(\psi_i(x, \phi_2(x)g_i)) = \psi_i(x, \phi_1(x)\phi_2(x)g_i). \qquad \square$$

Henceforth, when we have some gauge transformation Φ, then φ will always refer to the corresponding element of $C^\infty_{\mathrm{Ad}}(P, G)$ and $\{\phi_i\}_i$ will always be the element from $\Gamma^\infty(M, \operatorname{Ad} P)$.

Proposition 1.19 *Denote the center of a group by $Z(G)$. Then if the structure group G is Hausdorff, connected, compact and semisimple and if M is connected,*

$$Z(\mathscr{G}) \cong \Gamma^\infty(M, P \times_{\mathrm{Ad}} Z(G)) \cong Z(G). \tag{1.16}$$

Proof The first isomorphism follows readily from restricting the isomorphism $\mathscr{G} \cong \Gamma^\infty(M, \operatorname{Ad} P)$ to the center of $\mathscr{G}$. As to the second, if $\phi_i(x)$ is in the center of G for all i, then they satisfy $\phi_i(x) = g_{ij}(x)\phi_j(x)g_{ij}(x)^{-1} = \phi_j(x) =: \phi(x)$ for all x.

To finish the proof, we need that $Z(G)$ has the discrete topology. To see this, note that the center of a compact connected semisimple Lie group is finite (see Proposition C.3 in the appendix). A topological group G is Hausdorff if and only if $\{e\} \subset G$ is open, so if G is Hausdorff then $Z(G)$ is too. So $Z(G)$ is finite and Hausdorff, which can only be the case if it is discrete.

Now M is connected, so $\phi(M)$ is also connected. But the only connected subspaces of a discrete space are the empty set and the one-point sets. Thus ϕ cannot depend on its argument at all while being continuous, so $\phi \in Z(G)$. $\square$

Chapter 2

Connections

Here we deal with connections. A connection is an entity defined on the fiber bundle, which determines a physical configuration. We will see that there is a natural way in which a gauge transformation can act on a connection, transforming it into another connection which mathematically differs from the original, but which determines the same physical configuration. As shown in the previous chapters, principal bundles and vector bundles are closely related; in this chapter we will see that the same holds for connections on both of them.

We also discuss the concept of holonomy. Loosely speaking, if we have a path in the base space, then a connection provides us with a way of 'lifting' it to the fiber bundle, producing another path in the bundle. If we then take a path of which the begin and end coincide (a circle), then it will not necessarily lift to a circle in the fiber space. The end of such a lifted path relative to its begin provides us with information on the connection; this is called holonomy. Here we give some results that will be of use to us in Chapter 5 on moduli spaces.

Then we introduce characteristic classes. These are entities defined on a bundle that we will express in terms of a connection on that bundle, but they will actually turn out to be independent of the connection. Thus they depend only on the structure on the bundle. This allows one to see if two bundles differ from each other by comparing their characteristic classes. Moreover, they play various important roles in the physics that we deal with later on in this thesis.

More information on connections may be found in [4], [5, chapter 10], and [7].

2.1 Connections on principal bundles

Definition 2.1 Let $p \in P$. We call the kernel of the pushforward of the projection, $V_p := \ker(\pi_*|_p)$, the *vertical subspace* of T_pP. A tangent vector X at p is said to be *vertical* if $X \in V_p$. A vector field is vertical if its image is.

Proposition 2.2 *V_p is the tangent space of the fiber above $\pi(p)$. Thus, if $x = \pi(p)$, then $V_p = T_p(\pi^{-1}(x))$.*

Proof Suppose $X \in T_p(\pi^{-1}(x))$. Let $f : M \to \mathbb{R}$, then $\pi_* X(f) = X(f \circ \pi) = 0$, because $f \circ \pi$ is clearly constant when restricted to $\pi^{-1}(x)$. Thus, we have an injective inclusion $T_p(\pi^{-1}(x)) \subset V_p$. Now, $\pi^{-1}(x) \cong G$ so $\dim T_p(\pi^{-1}(x)) = \dim G$, while the dimension of $\ker \pi_*$ is also $\dim G$. Therefore, for dimensional reasons our inclusion is also a surjection. $\square$

Having one linear subspace of T_pP, we could try to find another subspace H_p so that we get a decomposition $T_pP = V_p \oplus H_p$. Indeed, this is one way of defining a connection on a principal bundle. Unlike V_p, however, there is no preferred H_p; in general there will be an infinite number of connections to choose from.

First, however, we need the following formalism.

2.1.1 Fundamental vector fields

Definition 2.3 Let $\mathcal{T}P$ be the space of vector fields on P. We define a map $\sigma : \mathfrak{g} \to \mathcal{T}P$ by

$$\sigma_p(X) = \frac{\mathrm{d}}{\mathrm{d}t}\, pe^{tX}\Big|_{t=0}. \tag{2.1}$$

$\sigma(X)$ is called the *fundamental vector field* of X. Note that when $t = 0$, then $pe^{tX} = p$, so $\sigma_p(X)$ is indeed a tangent vector at p. $\sigma(X)$, then, may be regarded as a vector field.

Proposition 2.4 *$\sigma_p(X)$ is a vector space isomorphism $\mathfrak{g} \cong V_p$ for each p. Furthermore, σ satisfies*

$$\begin{aligned}
&(R_g)_*\sigma_p(X) = \sigma_{pg}(\mathrm{ad}_{g^{-1}} X),\\
&\Phi_*\sigma_p(X) = \sigma_{\Phi(p)}(X),\\
&\sigma([X,Y]) = [\sigma(X), \sigma(Y)].
\end{aligned} \tag{2.2}$$

for any gauge transformation $\Phi \in \mathscr{G}$.

Note that σ_p is not a Lie algebra homomorphism, because its image, V_p, consists of tangent vectors which have no Lie bracket. On the other hand, $\sigma(X)$ (without the base point p) is a vector field, and vector fields do have Lie brackets.

Proof Define for each $p \in P$ a map $l_p : G \to P$ by $l_p(g) = pg$. Then

$$\sigma_p(X) = \frac{\mathrm{d}}{\mathrm{d}t} l_p(e^{tX})\bigg|_{t=0} = l_{p*}\frac{\mathrm{d}}{\mathrm{d}t}e^{tX}\bigg|_{t=0} = l_{p*}X,$$

in other words, σ_p is the pushforward of l_p at e. Therefore it is linear. Furthermore, if $x = \pi(p)$ then l_p is a diffeomorphism $G \to \pi^{-1}(x)$ since the group action is transitive and free. Therefore, the pushforward of l_p at e is a (vector space) isomorphism $l_{p*} : T_eG \xrightarrow{\sim} T_p(\pi^{-1}(x))$, or, switching notations, $\sigma_p : \mathfrak{g} \xrightarrow{\sim} V_p$. In particular, this shows that the image of σ is vertical.

For the first identity, we calculate

$$\begin{aligned}
(R_g)_*\sigma_p(X) &= \frac{\mathrm{d}}{\mathrm{d}t}\, R_g\left(pe^{tX}\right)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\left(pe^{tX}g\right)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\left(pgg^{-1}e^{tX}g\right)\Big|_{t=0}\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(pg\exp(t\,\mathrm{ad}_{g^{-1}} X)\right)\Big|_{t=0} = \sigma_{pg}(\mathrm{ad}_{g^{-1}} X).
\end{aligned}$$

As for the second,

$$\Phi_*\sigma_p(X) = \frac{\mathrm{d}}{\mathrm{d}t}\Phi(pe^{tX})\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\Phi(p)e^{tX}\bigg|_{t=0} = \sigma_{\Phi(p)}(X).$$

Lastly, write $R_g(p) = pg$. Recall that the Lie bracket of vector fields may be evaluated by $[X,Y] = \mathcal{L}_X Y$, with $\mathcal{L}$ the Lie derivative (see [6, p. 465]), and note that $\sigma(X)$ is generated by the flow $(t,p) \mapsto R_{\exp(tX)}(p)$. If $Y \in \mathfrak{g}$, we write Y_g for the value at g of its associated left-invariant vector field. Then

$$\begin{aligned}
[\sigma(X),\sigma(Y)]|_p &= \lim_{t\to 0} t^{-1}\left((R_{\exp(-tX)})_*\sigma_{p\exp(tX)}(Y) - \sigma_p(Y)\right)\\
&= \lim_{t\to 0} t^{-1}\left(\sigma_p(\mathrm{ad}_{\exp(tX)} Y) - \sigma_p(Y)\right)\\
&= \sigma_p\left(\lim_{t\to 0} t^{-1}\left(\mathrm{ad}_{\exp(tX)} Y - Y\right)\right) && (\sigma_p \text{ is linear})\\
&= \sigma_p\left(\lim_{t\to 0} t^{-1}\left((R_{\exp(tX)})_* Y_{\exp(-tX)} - Y\right)\right) && (Y \text{ is left-invariant})\\
&= \sigma_p([X,Y]).
\end{aligned}$$

□

2.1.2 The Maurer-Cartan form

Definition 2.5 The *Maurer-Cartan form* is defined by $\theta_g = (L_{g^{-1}})_* : T_g G \to T_e G$.

Proposition 2.6 *Let G be a matrix group, $G \subset GL(n)$, and let $g : G \hookrightarrow \mathrm{GL}(n)$ be the inclusion map. Then*

$$\theta_h = g(h)^{-1} \mathrm{d}g_h$$

where $\mathrm{d}g_h : T_h G \to T_{g(h)} \mathrm{GL}(n)$ is the pushforward at h of the inclusion g.[1]

Proof Take $X \in T_h G$, and let $\gamma : [-1, 1] \to G$ be a path such that $\gamma(0) = h$ and $\gamma'(0) = X$. Then, noting that $g(hh') = g(h)g(h')$,

$$\begin{aligned}(L_{h^{-1}})_* \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)\Big|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t} h^{-1}\gamma(t)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} g(h^{-1}\gamma(t))\Big|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} g(h^{-1}) g(\gamma(t))\Big|_{t=0} = g(h)^{-1} \frac{\mathrm{d}}{\mathrm{d}t} g \circ \gamma(t)\Big|_{t=0} \\ &= g(h)^{-1} \mathrm{d}g_h(X). \qquad \square\end{aligned}$$

We shall often write this as $g^{-1}\mathrm{d}g$. Similarly, if $f : M \to G$ is a function for some space M, then we shall also write $f^*\theta = f^{-1}\mathrm{d}f$. To see that this notation also makes sense, let $x \in M$ and $X \in T_x M$. Then

$$\begin{aligned}(f^*\theta)_x(X) &= \theta_{f(x)}(f_* X) = g(f(x))^{-1} \mathrm{d}g_{f(x)}(f_* X) = g(f(x)^{-1}) \mathrm{d}(g \circ f)_x(X) \\ &= f(x)^{-1} \mathrm{d}f_x(X).\end{aligned}$$

To the right of the second equality, we have the confusion of two notations $\mathrm{d}g_{f(x)}(f_* X) = g_*(f_*(X)) = (g \circ f)_*(X) = \mathrm{d}(g \circ f)_x(X)$.

The commutator for two $\mathfrak{g}$-valued forms is defined to be, if α is a p-form and β is a q-form, and if $\{T_a\}_a$ are the generators of $\mathfrak{g}$,

$$\begin{aligned}[\alpha, \beta] &= \alpha \wedge \beta - (-1)^{pq} \beta \wedge \alpha & (2.3) \\ &= T_a T_b \alpha^a \wedge \beta^b - (-1)^{pq} T_b T_a \beta^b \wedge \alpha^a \\ &= T_a T_b \alpha^a \wedge \beta^b - T_b T_a \alpha^a \wedge \beta^b = [T_a, T_b] \alpha^a \wedge \beta^b. & (2.4)\end{aligned}$$

In particular, note that if α is a p-form with p odd, then $\alpha \wedge \alpha = \frac{1}{2}[\alpha, \alpha]$.

Proposition 2.7 **The Maurer-Cartan equation.** *θ, when considered as a Lie algebra-valued one-form, satisfies*

$$\mathrm{d}\theta = -\tfrac{1}{2}[\theta, \theta]. \tag{2.5}$$

Proof Note that $0 = \mathrm{d}(e) = \mathrm{d}(g^{-1}g) = \mathrm{d}(g^{-1})g + g^{-1}\mathrm{d}g$, so $\mathrm{d}(g^{-1}) = -g^{-1}(\mathrm{d}g)g^{-1}$. Then

$$\mathrm{d}\theta = \mathrm{d}(g^{-1}) \wedge \mathrm{d}g = -g^{-1}(\mathrm{d}g)g^{-1} \wedge \mathrm{d}g = -(g^{-1}\mathrm{d}g) \wedge (g^{-1}\mathrm{d}g) = -\tfrac{1}{2}[\theta, \theta]. \qquad \square$$

2.1.3 Connections

Definition 2.8 With a *connection* we mean either of the following:

- An *Ehresmann connection* is a collection of linear subspaces $H_p \subset T_p P$ for each p, satisfying
  1. $T_p P$ decomposes as $T_p P = H_p \oplus V_p$,

[1] The notation for this particular pushforward differs from the one we usually use, which is an asterisk: g_*. This is because all of the existing literature seems to use $\mathrm{d}g$ in this case.

2. H_p depends smoothly on p, in the sense that if X is a vector field, then its pointwise decomposition $X_p = X_p^H + X_p^V$ consist of a horizontal and a vertical vector field which are both smooth,
3. $(R_g)_* H_p = H_{pg}$ (that is, H_p is equivariant).

- A *connection one-form* is a smooth form $\omega \in \Omega^1(P, \mathfrak{g}) := \Gamma^\infty(T^*P \otimes \mathfrak{g})$ (Γ^∞ being the space of smooth sections), satisfying
  1. $R_g^*\omega = \mathrm{ad}_{g^{-1}} \circ \omega$ for every g.
  2. $\omega(X) = \sigma^{-1}(X)$ if X is vertical.

Theorem 2.9 *These two types of connections are equivalent: an Ehresmann connection uniquely defines a connection one-form, and vice versa.*

Proof Recall that σ is an isomorphism, so that it is invertible. If we have an Ehresmann connection, then there is an associated projection onto the vertical subspace P^V. We write

$$\omega = \sigma^{-1} \circ P^V \quad \text{or equivalently} \quad P^V = \sigma \circ \omega. \tag{2.6}$$

If ω is a connection one-form, then we shall consider this equation to define a vertical projection P^V, and thereby a decomposition $T_pP = V_p \oplus H_p$. Conversely, if we have such a decomposition then will consider (2.6) to define a connection one-form.

Let ω be a connection one-form. Then $X^V := P^V(X) = \sigma \circ \omega(X)$ is a vertical vector field, because the image of σ is vertical. Note that P^V is indeed a projection in the sense that $(P^V)^2 = P^V$. We write $X^H = X - X^V$. Then we have found a direct sum $T_pP = H_p \oplus V_p$, $X = X^V + X^H$. Note that if X is a smooth vector field, then $X^V = \sigma \circ \omega(X)$ is a smooth vector field, because both σ and ω are smooth. Therefore, point 2 of the definition of an Ehresmann connection is satisfied. Lastly, let X be horizontal. This is equivalent with $0 = P^V(X) = \sigma \circ \omega(X)$, but since σ is an isomorphism, this is equivalent with $X \in \ker \omega$. Then $\omega((R_g)_* X) = R_g^*\omega(X) = \mathrm{ad}_{g^{-1}} \circ \omega(X) = 0$, so H_p also satisfies the third point of the definition of an Ehresmann connection.

Conversely, let $\{H_p\}_{p \in P}$ be an Ehresmann connection. We define a form $\omega = \sigma^{-1} \circ P^V$. This is then a smooth form by point 2 of the Ehresmann connection. Moreover, if $X = \sigma(Y)$ is vertical, then clearly $\omega(X) = \sigma^{-1} \circ P^V(X) = \sigma^{-1}(X) = Y$, so point 2 of the definition of a connection one-form is satisfied. It remains to be shown that $R_g^*\omega = \mathrm{ad}_{g^{-1}} \circ \omega$. Since ω is defined by its values on horizontal and vector fields, it suffices to check the identity only for these two cases. Let X be horizontal, so that $X \in H_p$ and $\omega(X) = \sigma^{-1} \circ P^V(X) = 0$. Then $R_g^*\omega(X) = \omega((R_g)_* X) = 0$, because $(R_g)_* X \in H_{pg}$. Now let X be vertical, so $X = \sigma_p(Y)$ for some Y. Then

$$\begin{aligned} R_g^*\omega(X) &= \omega\left((R_g)_*\sigma(Y)\right) = \omega\left(\sigma(\mathrm{ad}_{g^{-1}}(Y))\right) = \mathrm{ad}_{g^{-1}}(Y) = \mathrm{ad}_{g^{-1}}(\sigma^{-1}(X)) \\ &= \mathrm{ad}_{g^{-1}} \circ \omega(X). \end{aligned}$$

□

Proposition 2.10 *If ω is a connection one-form, then $H_p = \ker \omega_p$ if and only if H_p is the induced Ehresmann connection.*

Proof Let ω be a connection one-form and H_p its associated Ehresmann connection. Let X be arbitrary. Then $P^V(X) = \sigma(\omega(X)) = 0$ if and only if $\omega(X) = 0$ since σ is an isomorphism. Therefore $H_p = \ker \omega_p$.

Conversely, let ω be a connection one-form, and define the subspaces $H_p := \ker \omega_p$. This defines a vertical projection P^V. Take an arbitrary vector field X. Then

$$0 = \omega(X - X^V) = \omega(X) - \omega(X^V) = \omega(X) - \sigma^{-1} \circ P^V(X),$$

whence $\omega = \sigma^{-1} \circ P^V$, or equivalently $P^V = \sigma \circ \omega$. □

Example 2.11 If P has a G-invariant Riemannian metric, then one can define $H_p = V_p^\perp$. Indeed, suppose $X \in H_p$, and write an arbitrary element of V_{pg} as $\sigma_{pg}(Z)$ for some $Z \in \mathfrak{g}$. Then

$$g(R_{g*}X_p, \sigma_{pg}(Z)) = g(R_{g*}X_p, R_{g*}\sigma_p(\mathrm{ad}_g Z)) = g(X_p, \sigma_p(\mathrm{ad}_g Z)) = 0,$$

since $\sigma_p(\mathrm{ad}_g Z)$ is still vertical. It follows that $R_{g*}X_p \in H_{pg}$. The smoothness of this decomposition follows from that of g, so that this is indeed an Ehresmann connection.

Example 2.12 Let $P = M \times G$ be the trivial bundle, and write $p = (x, g)$. Then $V_p = T_{(x,g)}(\pi^{-1}(x)) = T_{(x,g)}(\{x\} \times G)$. The *flat connection* on P is $H_p := T_{(x,g)}(M \times \{g\})$. It is clear that T_pP decomposes into $T_pP = V_p \oplus H_p$. Moreover, R_g acts only on the second coordinate of $p = (x, g)$, which implies $(R_{g'})_*H_p = (R_{g'})_*T_{(x,g)}(M \times \{g\}) = T_{(x,gg')}(M \times \{gg'\}) = H_{pg'}$.

Theorem 2.13 *Any principal bundle P admits a connection.*

Proof On each trivialization $\pi^{-1}(U_i) = U_i \times G$, take the connection one-forms associated to the flat connections, and glue them together to all of P using a partition of unity subordinate to $\{U_i\}_i$. □

Flat connections will be treated in more detail on page 18.

Recall that a vector field X is said to be f-related to a vector field Y if $f_*X_x = Y_{f(x)}$ for all x. If X_i is f-related to Y_i for $i = 1, 2$, then $f_*[X_1, X_2]$ is f-related to $[Y_1, Y_2]$ [6, p. 92].

Proposition 2.14 *The space of smooth vertical vector fields is a Lie algebra-ideal in $\mathcal{T}P$. In other words, if X is any vector field and V is vertical, then $[X, V]$ is vertical.*

Proof If X, V are both vertical then $[X, V]$ is vertical by Proposition 2.4. Therefore, it suffices to prove the proposition for horizontal X. If X is horizontal, then we have $\pi_*X_{pg} = \pi_*R_{g*}X_p = \pi_*X_p$, because H_p is equivariant and because $\pi \circ R_g = \pi$. Since the action is transitive, this means that $\pi_*X_p = \pi_*X_{p'}$ for any p, p' which are in the same fiber. Thus the vector field π_*X is a well-defined vector field on M, and it is π-related to X. Furthermore, $\pi_*V_p = 0$ at any p, so V is obviously π-related to 0. Therefore,

$$\pi_*[X, V] = [\pi_*X, 0] = 0,$$

whence $[X, V]$ is vertical. □

Proposition 2.15 *If H is a horizontal vector field and V is a vertical vector field, then $[H, V]$ is horizontal.*

Proof Write $V = \sigma(X)$ for some $X \in \mathfrak{g}$. Then, using that that V is generated by the flow $(t, p) \mapsto R_{\exp(tX)}(p)$,

$$[V, H] = \mathcal{L}_V H = \lim_{t \to 0} t^{-1}((R_{\exp(-tX)})_*H - H),$$

with $\mathcal{L}$ the Lie derivative. But H is horizontal, so $(R_{\exp(tX)})_*H$ is also horizontal because the Ehresmann connection is equivariant, so $[H, V]$ is horizontal. □

Corollary 2.16 *If H is a horizontal vector field and V is a vertical vector field, then $[H, V] = 0$.*

Let X be a tangent vector on P. Then there is a horizontal projection h, projecting it into the horizontal subspace, i.e. $hX \in H_pP$. Now write h^* for its dual map on T^*P (note that this is *not* the pullback of a smooth map, despite the notation). Thus $(h^*\omega)(X_1, \ldots, X_k) := \omega(hX_1, \ldots, hX_k)$.

Definition 2.17 Let V be some vector space, and $\phi \in \Omega^k(P, V)$ and $X_1, \ldots, X_{k+1} \in T_pP$. Then the *covariant derivative* of ϕ is defined to be $D\phi := h^*(d_P\phi)$, i.e.

$$\mathrm{D}\phi(X_1, \ldots, X_{k+1}) = \mathrm{d}_P\phi(hX_1, \ldots, hX_{k+1}), \tag{2.7}$$

where d_P is the exterior derivative applied to the one-form-part of ϕ: $\mathrm{d}_P\phi = \mathrm{d}_P\phi^\alpha \otimes e_a$.

Definition 2.18 The *curvature two-form* Ω of a connection is the covariant derivative of the associated connection one-form ω:

$$\Omega = \mathrm{D}\omega \in \Omega^2(P, \mathfrak{g}). \tag{2.8}$$

Proposition 2.19 *The curvature two-form satisfies the following equations:*

$$\begin{aligned} &R_g^*\Omega = g^{-1}\Omega g, \\ &\Omega = \mathrm{d}_P\omega + \omega \wedge \omega, \\ &D\Omega = 0. \end{aligned} \tag{2.9}$$

The second equation may also be written as $\Omega(X, Y) = \mathrm{d}_P\omega(X, Y) + [\omega(X), \omega(Y)]$, as a little algebra will confirm.

Proof For the first equation,

$$\begin{aligned} R_g^*\Omega(X, Y) &= \mathrm{d}_P\omega(h(R_{g*}X), h(R_{g*}Y)) = \mathrm{d}_P\omega(R_{g*}(hX), R_{g*}(hY)) \\ &= R_g^*\mathrm{d}_P\omega(X^H, Y^H) = \mathrm{d}_P R_g^*\omega(X^H, Y^H) \\ &= \mathrm{d}_P(g^{-1}\omega g)(X^H, Y^H) = g^{-1}\Omega(X, Y)g. \end{aligned}$$

We shall prove the second equation for three separate cases.

- Let X, Y both be horizontal. Then $\omega(X) = \omega(Y) = 0$ so the second term on the right hand vanishes, while the left hand side becomes $\Omega(X, Y) = \mathrm{d}_P\omega(X^H, Y^H) = \mathrm{d}_P\omega(X, Y)$.
- Let X be horizontal and Y be vertical, so that there is a $V \in \mathfrak{g}$ such that $Y_p = \sigma_p(V)$. Then note that $\omega_p(Y_p) = V$ is constant. Also, $\omega(X) = 0$ and $[X, Y] = 0$, so

  $$\mathrm{d}_P\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]) = 0.$$

  Furthermore, $[\omega(X), \omega(Y)] = 0$ because X is horizontal. $\Omega(X, Y)$ vanishes too, because $\Omega(X, Y) = \mathrm{d}_P\omega(hX, hY)$ but $hY = 0$.
- Lastly, let X and Y be vertical, so there are $V, W \in \mathfrak{g}$ such that $X = \sigma(V)$ and $Y = \sigma(W)$. In this case, $\Omega(X, Y) = 0$, and similarly to the previous item, we find $\mathrm{d}_P\omega(X, Y) = -\omega([X, Y])$. But

  $$\omega([X, Y]) = \omega([\sigma(V), \sigma(W)]) = \omega(\sigma([V, W])) = [V, W] = [\omega(X), \omega(Y)],$$

  so $\mathrm{d}_P\omega(X, Y) + [\omega(X), \omega(Y)] = 0$ as required.

Since Ω is linear and antisymmetric, this suffices to prove the second equation.

Lastly, note that $(\mathrm{d}_P\Omega)^k = f_{ij}{}^k(\mathrm{d}_P\omega^i \wedge \omega^j + \omega^i \wedge \mathrm{d}_P\omega^j)$, with $f_{ij}{}^k$ the structure constants of $\mathfrak{g}$. Noting that $\omega(hX) = 0$ for any X, we then have $\mathrm{D}\Omega(X, Y, Z) = \mathrm{d}_P\Omega(hX, hY, hZ) = 0$. □

The gauge potential

If we have some section s_i on a chart U_i of M, then we can use this section to pull back ω to M.

Definition 2.20 The *gauge potential* of the connection is

$$A_i := s_i^*\omega \in \Omega^1(U_i, \mathfrak{g}). \tag{2.10}$$

Unless the bundle is trivial so that there exists a global section, this can only be done on local patches. If we have sections s_i for each patch of an open covering $\{U_i\}$ of M, then ω gives the gauge potentials $\{A_i\}$.

Conversely, as the next theorem shows, an A_i also locally determines an ω. Write a local trivialization as $\psi_i^{-1}(p) = (x, g_i(p))$. Any section s_i has a canonical trivialization ψ_i associated to it; when written this way, we have $p = \psi_i(x, g_i(p)) = \psi_i(x, e)g_i(p) = s_i(x)g_i(p)$.

Proposition 2.21 *Given a $\mathfrak{g}$-valued one-form A_i on a trivialization U_i, and a local section $s_i : U_i \to P$, there is a connection one-form ω_i on $\pi^{-1}(U_i)$ such that $A_i = s_i^*\omega_i$, given by*

$$\omega_i = g_i^{-1}\pi^* A_i g_i + g_i^{-1}\mathrm{d}g_i, \tag{2.11}$$

where g_i is as above.

Proof Let $x \in M$, then

$$\begin{aligned}(s_i^*\omega_i)_x(X) &= \omega_i|_{s_i(x)}((s_i)_*X) \\ &= g_i(s_i(x))^{-1}\, A_i|_x(\pi_*(s_i)_*X)g(s_i(x)) + g_i^*\theta((s_i)_*X)\end{aligned}$$

The second term can also be written as $\theta((g_i \circ s_i)_*X)$. But $g_i(s_i(x)) = g_i(\psi_i(x,e)) = e$, so $(g_i \circ s_i)_* = 0$. Therefore the second term vanishes, while the remaining g_i disappear from the formula:

$$s_i^*\omega_i(X) = A_i(\pi_*(s_i)_*X) = A_i(X)$$

as required. It remains to be shown that ω_i is indeed a connection one-form.

- Suppose X is vertical, so $X_p = \sigma_p(Y)$ for some $Y \in \mathfrak{g}$. Note that $\pi_*X = 0$, so the first term of ω_i vanishes. The second term may be written $g_i^{-1}\mathrm{d}g_i = g_i^*\theta$, so

$$\begin{aligned}(\omega_i)_p(X_p) &= (g_i^*\theta)_p\left(\frac{\mathrm{d}}{\mathrm{d}t}pe^{tY}\bigg|_{t=0}\right) = \theta_{g_i(p)}\left((g_i)_*\,\frac{\mathrm{d}}{\mathrm{d}t}pe^{tY}\bigg|_{t=0}\right) \\ &= (L_{g_i(p)^{-1}})_*\left(\frac{\mathrm{d}}{\mathrm{d}t}g_i(p)e^{tY}\bigg|_{t=0}\right) = \frac{\mathrm{d}}{\mathrm{d}t}g_i(p)^{-1}g_i(p)e^{tY}\bigg|_{t=0} = Y.\end{aligned}$$

- We have

$$\begin{aligned}(R_g^*\omega_i)_p(X_p) &= \omega_{pg}((R_g)_*X_p) \\ &= g_i(pg)^{-1}A_i(\pi_*(R_g)_*X_p)g_i(pg) + g_i(pg)^{-1}\mathrm{d}g_i|_{pg}\end{aligned}$$

  But $\pi_*(R_g)_* = (\pi \circ R_g)_* = \pi_*$, and $g_i(pg) = g_i(p)g$. Therefore $R_g^*\omega_i = \mathrm{Ad}_{g^{-1}}\,\omega_i$ as required.

This completes the proof. □

Proposition 2.22 *Suppose we have a set of gauge potentials $\{A_j\}$ and sections $\{s_j\}$ on an open covering $\{U_j\}$ of M. Denote the connection one-forms that they locally induce by $\{\omega_j\}$. Then $\omega_i = \omega_j$ for all i, j if and only if A_i and A_j satisfy for all i, j*

$$A_j = g_{ij}^{-1}A_i g_{ij} + g_{ij}^{-1}\mathrm{d}g_{ij}. \tag{2.12}$$

Proof Take the canonical sections s_i, s_j associated to the trivializations. Note that the functions g_i satisfy $g_i(s_j(x)) = g_i(\psi_j(x,e)) = g_i(\psi_i(x, g_{ij}(x))) = g_{ij}(x)$. Suppose $\omega_i = \omega_j$. We calculate

$$\begin{aligned}A_j|_x(X) = (s_j^*\omega_j)_x(X) &= (s_j^*\omega_i)_x(X) = \omega_i|_{s_j(x)}(s_{j*}X) \\ &= g_i(s_j(x))^{-1}(\pi^*A_i)_{s_j(x)}(s_{j*}X)g_i(s_j(x)) + (g_i^*\theta)_{s_j(x)}(s_{j*}X) \\ &= g_i(s_j(x))^{-1}\, A_i|_{\pi(s_j(x))}(\pi_* s_{j*}X)g_i(s_j(x)) + ((g_i \circ s_j)^*\theta)_x(X) \\ &= g_{ij}(x)^{-1}\, A_i|_x(X)g_{ij}(x) + g_{ij}(x)^{-1}\,\mathrm{d}g_{ij}|_x(X),\end{aligned}$$

where we used that $\pi \circ s_j(x) = x$.

Conversely, the calculation above shows that if A_j and A_j are related on the overlap by formula (2.12), then $s_j^*\omega_j = s_j^*\omega_i$. But we have by definition of ω_j that $\omega_j = g_i^{-1}\pi^*s_j^*\omega_j g_j + g_j^{-1}\mathrm{d}g_j$. In other words, the value of ω_j is entirely determined by the pullback $s_j^*\omega_j$. Therefore, $\omega_j = \omega_i$. □

Thus, giving a set of gauge potentials which satisfy equation (2.12) is a third way to specify a connection. We also see that if ω, ω' are two connections, then we have $\tau_j := A_j - A'_j = g_{ij}^{-1}(A_i - A'_i)g_{ij}$ on overlaps. Thus, τ_j is a family of sections of $\Omega^1(U_j, \mathfrak{g})$ satisfying the condition $\tau_j = \mathrm{ad}(g_{ji})\tau_i$, so they define a one-form with values in the adjoint bundle $\mathrm{ad}\, P$. So the space of connections $\mathscr{A}$ is an affine space modeled on $\Omega^1(M, \mathrm{ad}\, P)$.

Note that a one-form $\omega \in \Omega^1(P, \mathfrak{g})$ has to satisfy two demands in order to be a connection one-form, while *any* one-form $A \in \Omega^1(U, \mathfrak{g})$ *locally* determines a connection. On the other hand, if we want to have a connection on all of M, however, then we must have multiple one-forms $\{A_i\}_i$ satisfying (2.12) on overlaps. However, on multiple occasions we shall encounter bundles over $\mathbb{R}^4$, which are trivial. In this case this demand is trivially satisfied.

Definition 2.23 The *field strength* F_i is the local form of Ω:

$$F_i = s_i^*\Omega. \tag{2.13}$$

Proposition 2.24 *Like A_i, the field strength satisfies a compatibility condition on the overlap of two charts U_i and U_j:*

$$F_j = g_{ij}^{-1} F_i g_{ij}. \tag{2.14}$$

In addition, it satisfies the following equations (assuming that F and A are pulled back using the same sections, and dropping the subscripts):

$$\begin{aligned} &F = \mathrm{d}A + A \wedge A, \\ &\mathcal{D}_A F := \mathrm{d}F + [A, F] = 0. \end{aligned} \tag{2.15}$$

The operator $\mathcal{D}_A = \mathrm{d} + [A, \cdot]$ is also called the *covariant derivative*. The second identity is called the Bianchi identity.

Proof We fist prove the second identity:

$$F = s^*\Omega = s^*(\mathrm{d}_P\omega + \omega \wedge \omega) = \mathrm{d}(s^*\omega) + (s^*\omega) \wedge (s^*\omega) = \mathrm{d}A + A \wedge A.$$

The third is also easy:

$$\begin{aligned} \mathrm{d}F &= \mathrm{d}(s^*\Omega) = s^*(\mathrm{d}_P\Omega) = s^*(\mathrm{d}_P\omega \wedge \omega - \omega \wedge \mathrm{d}_P\omega) = \mathrm{d}A \wedge A - A \wedge \mathrm{d}A \\ &= F \wedge A - A \wedge F = -[A, F]. \end{aligned}$$

The first identity then follows immediately from substituting $A_j = g_{ij}^{-1} A_i g_{ij} + g_{ij}^{-1}\mathrm{d}g_{ij}$ in the second, expanding, and canceling terms. □

Proposition 2.25 *The covariant derivative satisfies*

$$\mathcal{D}_A^2 = [F, \cdot]. \tag{2.16}$$

Proof Let $\xi \in \Omega^p(U, \mathfrak{g})$. Then

$$\mathcal{D}_A^2\xi = \mathrm{d}(\mathcal{D}_A\xi) + [A, \mathcal{D}_A\xi] = \mathrm{d}[A, \xi] + [A, \mathrm{d}\xi] + [A, [A, \xi]].$$

Now,

$$\begin{aligned} \mathrm{d}[A, \xi] &= [T_a, T_b]\mathrm{d}(A^a \wedge \xi^b) = [T_a, T_b](\mathrm{d}A^a \wedge \xi^b + (-1)^1 A^a \wedge \mathrm{d}\xi^b) \\ &= [\mathrm{d}A, \xi] - [A, \mathrm{d}\xi], \end{aligned}$$

and

$$[A, [A, \xi]] = [A, A \wedge \xi - (-1)^p \xi \wedge A]$$

$$
\begin{aligned}
&= A \wedge A \wedge \xi - (-1)^p A \wedge \xi \wedge A \\
&\quad - (-1)^{p+1}(A \wedge \xi \wedge A - (-1)^p \xi \wedge A \wedge A) \\
&= A \wedge A \wedge \xi + (-1)^{2p+1} \xi \wedge A \wedge A \\
&= [A \wedge A, \xi].
\end{aligned}
$$

Thus

$$\mathcal{D}_A^2 \xi = [\mathrm{d}A, \xi] + [A \wedge A, \xi] = [F, \xi]. \qquad \square$$

Proposition 2.26 *Let s, s' be two sections on a local trivialization U such that they are related by a map $\phi : U \to G$ by $s'(x) = s(x)\phi(x)$. (This can be done for any two smooth local sections.) By pulling back a connection one-form ω, either section defines a gauge potential, which are related by*

$$A' = \phi^{-1} A \phi + \phi^{-1} \mathrm{d}\phi. \tag{2.17}$$

Proof As noted before, s and s' both define trivializations ψ, ψ' of which they are the canonical sections. Let g denote the transition function between these two trivializations. Then

$$s'(x) = \psi'(x, e) = \psi(x, g(x)e) = \psi(x, eg(x)) = \psi(x, e)g(x) = s(x)g(x).$$

Since the group action is free, this means that $g(x) = \phi(x)$, so ϕ takes on the role of a transition function. Then the statement follows from (2.12). $\square$

Gauge transformations acting on connections

Recall that a gauge transformation induces a function $\phi : U \to G$ on the trivialization U, as in the second equality of theorem 1.18. Therefore, we make the following definition.

Definition 2.27 Let $\Phi \in \mathscr{G}$ be a gauge transformation, and let $\phi : U \to G$ the function that it induces on a local trivialization U. Then we define an action of $\mathscr{G}$ on the space of connections by the previous proposition, i.e.

$$\Phi(A) = \phi^{-1} A \phi + \phi^{-1} \mathrm{d}\phi. \tag{2.18}$$

Proposition 2.28 *Let $\{A_i\}_i$ be a connection, and let ω and H be its connection one-form and Ehresmann connection, respectively. Let $\Phi \in \mathscr{G}$ be a gauge transformation, and write $\Phi(\omega)$ and H^Φ for the connection one-form and the Ehresmann connection gauge transformed along Φ. Then we have*

$$\Phi(\omega) = \Phi^* \omega \tag{2.19}$$

and

$$H_p^\Phi = \Phi_*^{-1} H_{\Phi(p)}. \tag{2.20}$$

Proof We first show that H^Φ and $\Phi(\omega)$ are well-defined as connections, and that they are equivalent to each other. Then we will show that $\Phi(\omega)$ is equivalent with $\Phi(A)$.

Note that Φ preserves vertical vector fields by Proposition 2.4. Furthermore, since Φ^{-1} is also a gauge transformation,

$$
\begin{aligned}
(R_g)_* H_p^\Phi &= (R_g)_* \Phi_*^{-1} H_{\Phi(p)} = \Phi_*^{-1} (R_g)_* H_{\Phi(p)} && \text{(equivariance of } \Phi^{-1}) \\
&= \Phi_*^{-1} H_{\Phi(p)g} && \text{(equivariance of } H_p) \\
&= \Phi_*^{-1} H_{\Phi(pg)} && \text{(equivariance of } \Phi) \\
&= H_{pg}^\Phi
\end{aligned}
$$

Thus, H^Φ is indeed an Ehresmann connection.

If ω is a connection one-form, then using similar arguments, it is easy to see that $\Phi^*\omega$ still satisfies $R_g^*\Phi^*\omega = \mathrm{Ad}_{g^{-1}}\Phi^*\omega$. Moreover, suppose X is vertical so that $X = \sigma(Y)$ for some $Y \in \mathfrak{g}$, then

$$\Phi^*\omega(X) = \omega(\Phi_*\sigma(Y)) = \omega(\sigma(Y)) = Y = \sigma^{-1}(X)$$

by Proposition 2.4. Thus, ω is indeed a connection one-form.

Now write $q = \Phi(p)$, and suppose $X_p \in H_p^\Phi = \Phi_*^{-1}H_q$. Thus there is a $Y_q \in H_q$ such that $X_p = \Phi_*^{-1}Y_q$. Then

$$(\Phi^*\omega)_p(X_p) = \omega_q(\Phi_*X_p) = \omega_q(\Phi_*\Phi_*^{-1}Y_q) = 0,$$

so $\Phi^*\omega$ is indeed the connection one-form associated to H^Φ. All we have to show now is that either of them are equivalent to $\Phi(A)$.

Consider a section s along which we will pull back ω, and consider its canonical trivialization $\psi(x,g) := s(x)g$. If Φ is a gauge transformation, then $\Phi\circ s(x) = \Phi(\psi(x,e)) = \psi(x,\phi(x)e) = \psi(x,e)\phi(x) = s(x)\phi(x)$. Thus $\Phi\circ s$ defines another section s' which is related to s by $s'(x) = s(x)\phi(x)$, so

$$s^*\Phi(\omega) = s^*\Phi^*\omega = (\Phi\circ s)^*\omega = \phi^{-1}A\phi + \phi^{-1}\mathrm{d}\phi$$

by Proposition 2.26. □

Flat connections

Now let $P = M\times G$ be the trivial bundle, and write $p = (x,g)$.

Definition 2.29 The *canonical flat connection* is defined to be $H_p = T_p(M\times\{g\})$. Equivalently, if $q : M\times G\to G$ is the projection onto the second factor, then $\omega = q^*\theta$.

To see the equivalence, note that if $p = (x,g)$ then $T_pP \cong T_xM\oplus T_gG$. Thus any tangent vector $X\in T_pP$ can be written as $X^i\frac{\partial}{\partial x^i} + X^j\frac{\partial}{\partial g^j}$. Clearly, then, $V_p = \ker\pi_* = \{0 + X^j\frac{\partial}{\partial g^j}\}$. Moreover, note that $\omega = q^*\theta = \theta\circ q_* = (L_{g^{-1}})_*\circ q_* = (L_{g^{-1}}\circ q)_*$. Now,

1. Suppose X is horizontal, so $X = X^i\frac{\partial}{\partial x^i} + 0$. Then $(q^*\theta)(X) = \theta(q_*(X)) = \theta(0) = 0$.
2. Suppose X is vertical, so $X = 0 + X^j\frac{\partial}{\partial g^j}$. Then $X = \sigma_p(Y)$ for some $Y\in\mathfrak{g}$, and writing $p = (x,g)$ again,

$$\begin{aligned} q^*\theta(\sigma_p(Y)) &= (L_{g^{-1}}\circ q)_*\left(\frac{\mathrm{d}}{\mathrm{d}t}\,pe^{tY}\Big|_{t=0}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\,(L_{g^{-1}}\circ q)(pe^{tY})\Big|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\,L_{g^{-1}}(ge^{tY})\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\,e^{tY}\Big|_{t=0} = Y. \end{aligned}$$

Thus $q^*\theta$ is in fact a connection one-form, and it is the one that is equivalent to $H_p = T_p(M\times\{g\})$.

Definition 2.30 A connection on a (nontrivial) bundle is said to be *flat* if it is locally isomorphic to the canonical flat connection. In other words, the connection is flat if there exists a diffeomorphism $\phi : \pi^{-1}(U)\to U\times G$ which maps the horizontal subspace at $u\in\pi^{-1}(U)$ upon the horizontal subspace at $\phi(U)$ of the canonical flat connection on $U\times G$.

Theorem 2.31 *A connection on a bundle is flat if and only if any of the following equivalent statements hold:*

1. $\Omega = 0$.
2. *If X, Y are two horizontal vectors, then $\omega([X,Y]) = 0$.*

3. *For each $p \in P$, the horizontal subspace $H_p \subset T_pP$ is closed under the Lie bracket: $X, Y \in H_p \Rightarrow [X, Y] \in H_p$.*
4. *$F_A = 0$ on every trivialization.*

Proof We will first show that a flat connection is equivalent to point 1, and then that all the statements are equivalent.

Suppose that we have a flat connection one-form $\omega = q^*\theta$ on a trivial bundle $U \times G$. Then

$$\mathrm{d}\omega = \mathrm{d}(q^*\theta) = q^*\mathrm{d}\theta = q^*(-\tfrac{1}{2}[\theta,\theta]) = -\tfrac{1}{2}[q^*\theta, q^*\theta] = -\tfrac{1}{2}[\omega,\omega] = -\omega \wedge \omega$$

so $\Omega = 0$. The other way around follows from the Ambrose-Singer theorem; we will prove this immediately below Theorem 2.40.

$1 \Leftrightarrow 2$:

$$\begin{aligned}\Omega(X,Y) &= \mathrm{d}_P\omega(X^H, Y^H) = X^H\omega(Y^H) - Y^H\omega(X^H) - \omega([X^H, Y^H]) \\ &= -\omega([X^H, Y^H]).\end{aligned}$$

If $\Omega = 0$ then the left hand side vanishes identically, so point 2 follows. On the other hand, if $\omega([X^H, Y^H]) = 0$ for any horizontal X^H, Y^H, then point 1 follows.

Point 2 is obviously equivalent with point 3 by $H = \ker(\omega)$.

$1 \Rightarrow 4$ is trivial. As to the other way around, since we are working on a trivialization, we may assume that the bundle is trivial, $P = U \times G$. Note that since the horizontal projection is idempotent, we have $\Omega(X,Y) = \Omega(X^H, Y^H)$. This implies that if one of its arguments is vertical, then Ω gives zero. Now let H_p be our connection and $T_pP = H'_p \oplus V_p$ any other decomposition. Then it defines a projection $T_pP \to H'_p$, $X \mapsto X^{H'}$ as well (different from that of the connection H_p), and $X = X^{H'} + X^{V'}$. Then

$$\Omega(X,Y) = \Omega(X^{H'} + X^{V'}, Y) = \Omega(X^{H'}, Y) + \Omega(X^{V'}, Y) = \Omega(X^{H'}, Y)$$

while on the other hand $\Omega(X,Y) = \Omega(X^H, Y)$. Thus Ω does not depend on the way in which we decompose T_pP. (Note, however, that we are *not* changing the connection from H_p to H'_p. If we did, then ω would change, so Ω would change. The only thing that we are doing is decomposing the arguments of Ω in a way different from the connection H_p.)

Now, let $s : U \to P$ be a section. Then note that π is a smooth left-inverse of s, so that π_* is a smooth left-inverse of s_*, whence s is an immersion. Therefore, $\ker(s \circ \pi)_* = \ker \pi_* = V_p$, so we have found a decomposition $T_pU = \mathrm{im}(s \circ \pi)_* \oplus V_p$. Let us write $X = X^s + X^V$ for this decomposition. Then, pulling back Ω along this section to obtain F_A,

$$\begin{aligned}\Omega(X,Y) &= \Omega(X^s, Y^s) = \Omega(s_*\pi_*X, s_*\pi_*Y) = s^*\Omega(\pi_*X, \pi_*Y) = F_A(\pi_*X, \pi_*Y) \\ &= 0.\end{aligned}$$

(Note that if pulling back Ω along one local section gives zero, then Ω gives zero when pulled back along *any* local section by Proposition 2.24.) □

Corollary 2.32 *Locally, $F_A = 0$ if and only if A is of the form $g^{-1}\mathrm{d}g$ for $g : U \to G$.*

Proof Since we are working locally, we may assume that the bundle is trivial, $\pi^{-1}(U) \cong U \times G$. Suppose $A = g^{-1}\mathrm{d}g$, i.e. it is $A' = 0$ gauge transformed along g. Then $0 = F_{A'} = g^{-1}F_A g$, so $F_A = 0$.

As to the other way around, let $q : U \times G \to G$ be the projection onto the second factor from Definition 2.29. Let $s : U \to P$ be a section, and define $g := q \circ s$. This is then a function from M to G. Now $F_A = 0$ implies that the connection is locally flat, so that $\omega = q^*\theta$, so we get for A by the remark below Proposition 2.6

$$A = s^*q^*\theta = (q \circ s)^*\theta = g^*\theta = g^{-1}\mathrm{d}g.$$ □

2.1.4 Holonomy

Let $P(M,G)$ be a principal bundle. Suppose $\gamma : [0,1] \to M$ is a smooth curve. Then for any $p \in \pi^{-1}(\gamma(0))$, there exists a unique smooth curve $\tilde{\gamma}_p : [0,1] \to P$ which has $\tilde{\gamma}_p(0) = p$, $\pi \circ \tilde{\gamma}_p = \gamma$, and is horizontal (i.e. $\dot{\tilde{\gamma}}_p(t) \in H(\gamma(t))$ for all t, or equivalently, $\omega(\dot{\tilde{\gamma}}_p(t)) = 0$). The curve $\tilde{\gamma}_p$ is called a *horizontal lift* of γ.

Proposition 2.33 *Let $\tilde{\gamma}'$ be another horizontal lift of γ, and let g be such that $\tilde{\gamma}'(0) = \tilde{\gamma}(0)g$. Then $\tilde{\gamma}'(t) = \tilde{\gamma}(t)g$ for all $t \in [0,1]$.*

Proof If $\tilde{\gamma}(t)$ is horizontal for all t, then $\tilde{\gamma}(t)g$ is horizontal as well: $\tilde{\gamma}(t)g = (R_g \circ \tilde{\gamma})(t)$, so $\frac{\mathrm{d}}{\mathrm{d}t}(R_g \circ \tilde{\gamma}) = (R_g)_* \dot{\tilde{\gamma}} \in H_{\tilde{\gamma}g}$. $\tilde{\gamma}(t)g$ obviously has $\tilde{\gamma}(0)g$ as starting point, so it must coincide with $\tilde{\gamma}'$ for all t. □

If we have such a γ and p, then they define a point $\tilde{\gamma}_p(1) \in \pi^{-1}(\gamma(1))$, called the *parallel transport* of p along γ. This means that any γ defines a map

$$\tau_\gamma : \pi^{-1}(\gamma(0)) \to \pi^{-1}(\gamma(1)), \quad p \mapsto \tau_\gamma(p) := \tilde{\gamma}_p(1). \tag{2.21}$$

Proposition 2.34 *This map is equivariant; $R_g \tau_\gamma = \tau_\gamma R_g$.*

Proof We have $\tau_\gamma(pg) = \tilde{\gamma}_{pg}(1)$. The lift $\tilde{\gamma}_{pg}(1)$ starts at $pg = \tilde{\gamma}(0)g$, so using Proposition 2.33

$$\tau_\gamma(pg) = \tilde{\gamma}_{pg}(1) = \tilde{\gamma}_p(1)g = \tau_\gamma(p)g.$$ □

Now suppose that γ is a loop, $\gamma(0) = \gamma(1) = x$. Then τ_γ is a map $\pi^{-1}(x) \to \pi^{-1}(x)$, which is equivariant; $\tau_\gamma(pg) = \tau_\gamma(p)g$. In general, for some loop γ we will have $\tilde{\gamma}_p(0) \neq \tilde{\gamma}_p(1)$, or equivalently, $p \neq \tau_\gamma(p)$. Then there exists a $g_{p,\gamma} \in G$ such that $\tau_\gamma(p) = pg_{p,\gamma}$. If, however, the connection is flat, $H_p = T_xM \oplus 0$ for $p = (x,g)$, then $\tilde{\gamma}(t) = (\gamma(t), g)$ is the horizontal lift of γ starting at (x,g). This lift also ends at (x,g). Thus, the set of these $g_{p,\gamma}$'s is a measure for the non-flatness of the connection.

Definition 2.35 The *holonomy group* at p of a connection ω is a subgroup of G defined by

$$\mathrm{Hol}_p(\omega) = \{g_{p,\gamma} \in G \mid \tau_\gamma(p) = pg_{p,\gamma}, \gamma : [0,1] \to M, \gamma(0) = \gamma(1) = \pi(p)\}. \tag{2.22}$$

The *restricted holonomy group* at p is the subgroup $\mathrm{Hol}^0_p(\omega)$ coming from horizontal lifts of contractible loops.

If α and β are two paths, then one may define another path $\alpha * \beta$ by going first through α and then through β. Then, denoting $\tau_\alpha(p) = pg_\alpha$ and similar for β and $\gamma := \alpha * \beta$, we have

$$\tau_\gamma(p) = \tau_\beta \circ \tau_\alpha(p) = \tau_\beta(pg_\alpha) = \tau_\beta(p)g_\alpha = pg_\beta g_\alpha, \tag{2.23}$$

which shows that $g_\gamma = g_\beta g_\alpha$ so that $\mathrm{Hol}_p(\omega)$ is indeed a group.

Proposition 2.36 *Let $P(M,G)$ be a principal bundle, $p \in P$, $g \in G$ and ω a connection.*
- $\mathrm{Hol}_{pg}(\omega) = g^{-1}\,\mathrm{Hol}_p(\omega)g$.
- *Let $q \in P$ be such that it is connected to p via a horizontal curve γ. Then* $\mathrm{Hol}_p(\omega) = \mathrm{Hol}_q(\omega)$.

Proof For the first item, take a $g_{p,\gamma} \in \mathrm{Hol}_p(\omega)$ for some path γ, so that $\tau_\gamma(p) = pg_{p,\gamma}$. Then $\tau_\gamma(pg) = \tau_\gamma(p)g = pg_{p,\gamma}g = pg(g^{-1}g_{p,\gamma}g)$, so that $g^{-1}g_{p,\gamma}g \in \mathrm{Hol}_{pg}(\omega)$. It is also easy to see that there is an inclusion the other way around, completing the proof of the first point.

As to the second, define an equivalence relation $p \sim q$ by demanding that there is a horizontal curve γ connecting p and q, i.e. $\gamma(0) = p$ and $\gamma(1) = q$. It is clear that this is indeed an equivalence relation. Note that $g \in \mathrm{Hol}_p(\omega)$ if and only if $p \sim pg$. Moreover, if γ is the horizontal path realizing $q \sim p$, then multiplying γ on the left by any $g \in G$ is a horizontal path connecting pg and qg. Therefore, $p \sim q$ implies $pg \sim qg$ for any g.

Now suppose $p \sim q$ and $g \in \mathrm{Hol}_p(\omega)$. Then on the one hand $pg \sim p \sim q$. On the other hand, $pg \sim qg$. By transitiveness of $\sim$, then, we find $qg \sim q$, i.e. $g \in \mathrm{Hol}_q(\omega)$. Since this argument is symmetric in p and q, it follows that $\mathrm{Hol}_p(\omega) = \mathrm{Hol}_q(\omega)$. □

Corollary 2.37 *Let ω be a connection. If M is connected, then for all $p, q \in P$ we have $\mathrm{Hol}_p(\omega) = \mathrm{Hol}_q(\omega)$.*

Proof A connected manifold is also path connected. Taking a path γ from $\pi(p)$ to $\pi(q)$ and lifting it, one obtains a path $\tilde{\gamma}_p$ starting at p and ending in the fiber of q. That is, $p \sim qg$ for some g, where $\sim$ is the equivalence relation introduced in the proof of the previous proposition. The statement then follows from the previous proposition. □

Theorem 2.38 *Let $P(M, G)$ be a principal bundle, and $p \in P$.*

- *$\mathrm{Hol}^0_p(\omega)$ is the identity component of $\mathrm{Hol}_p(\omega)$.*
- *If M is simply connected, then $\mathrm{Hol}^0_p(\omega) = \mathrm{Hol}_p(\omega)$.*
- *The connection is flat if and only if $\mathrm{Hol}^0_p(\omega)$ is trivial.*

Sometimes, the holonomy of a connection is defined in a different way. Take a path γ and a gauge transformation Φ. Both of them have associated group elements $g_{p,\gamma}$ and $\varphi(p)$ that multiply p on the right. On the other hand, for Φ there is another description, namely the functions $\phi_i(x)$ that multiply the G-coordinate of a point on the left. Inspired by this, if ψ_i is a local trivialization and $p = \psi_i(x, g_i)$, we set

$$\tau_\gamma(p) = \psi_i(x, g^i_\gamma g_i). \tag{2.24}$$

Thus we get another group, $\mathrm{Hol}^{\mathrm{L}}_x(\omega)$ say, where the L indicates the multiplication on the left.

Proposition 2.39 *The group $\mathrm{Hol}^{\mathrm{L}}_x(\omega)$ has the following properties.*

1. *It depends only on the base point $\pi(p) = x \in M$ instead of on p, but it does depend on the trivialization ψ_i.*
2. *For each $p = \psi_i(x, g_i)$, $\mathrm{Hol}^{\mathrm{L}}_x(\omega)$ is isomorphic to $\mathrm{Hol}_p(\omega)$ by*

$$\mathrm{Hol}^{\mathrm{L}}_x(\omega) = g_i \,\mathrm{Hol}_p(\omega) g_i^{-1}. \tag{2.25}$$

Proof Fix a path γ and point $p = \psi_i(x, g_i)$. Then any other point $p' \in \pi^{-1}(x)$ is given by $p' = pg$ for some g. For the moment, we write $g^i_\gamma(p)$ and $g^i_\gamma(pg)$ for the associated elements of $\mathrm{Hol}^{\mathrm{L}}_x(\omega)$. Then on the one hand $\tau_\gamma(pg) = \psi_i(x, g^i_\gamma(pg) g_i g)$, while on the other hand

$$\tau_\gamma(pg) = \tau_\gamma(p) g = \psi_i(x, g^i_\gamma(p) g_i) g = \psi_i(x, g^i_\gamma(p) g_i g).$$

Therefore, $g^i_\gamma(p) = g^i_\gamma(pg)$. Since the group action is transitive and free, this implies that g^i_γ only depends on the base point x of γ, so that we are allowed to write g^i_γ and $\mathrm{Hol}^{\mathrm{L}}_x(\omega)$.

If ψ_j is another trivialization overlapping with ψ_i, then

$$\begin{aligned}\tau_\gamma(p) &= \psi_i(x, g^i_\gamma g_i) = \psi_j(x, g_{ji}(x) g^i_\gamma g_i) \\ &= \psi_j(x, g_{ji}(x) g^i_\gamma g_{ij}(x) g_{ji}(x) g_i) \\ &= \psi_j(x, g_{ji}(x) g^i_\gamma g_{ij}(x) g_j),\end{aligned}$$

which implies $g^j_\gamma = g_{ij}(x)^{-1} g^i_\gamma g_{ij}(x)$. This proves the first point.

As to the second point, we have $\tau_\gamma(p) = \psi_i(x, g^i_\gamma g_i) = \psi_i(x, g_i g_{p,\gamma})$. This implies $g^i_\gamma g_i = g_i g_{p,\gamma}$, which implies $g^i_\gamma = g_i g_{p,\gamma} g_i^{-1}$. □

Theorem 2.40 **Ambrose-Singer.** *The Lie algebra of $\mathrm{Hol}_p(\omega)$ is spanned by all elements of the form*

$$\Omega_q(X, Y)$$

where q ranges over all points which can be joined to p by a horizontal curve, and $X, Y \in H_q$.

(For a proof, see [8]). With this theorem we can prove that $\Omega = 0$ implies that the connection is flat: If $\Omega = 0$, then the span of $\Omega_q(X, Y)$ and thus the Lie algebra of the holonomy is zero-dimensional. Therefore, the holonomy group is trivial, so $\mathrm{Hol}^0(\omega)$ is trivial, so the connection is flat by the last point of theorem 2.38.

Definition 2.41 A connection ω on a connected manifold is said to be *irreducible* if $\mathrm{Hol}(\omega) = G$.

As explained before, a gauge transformation $\Phi : P \to P$ may be viewed either as acting only on the gauge potential, or as acting on the connection itself. In the latter point of view, it also changes the parallel transport. For the moment, let us make explicit the dependence of the parallel transport on the connection by writing τ^ω_γ.

Lemma 2.42 *Let $\Phi \in \mathscr{G}$, then*

$$\Phi \circ \tau^\omega_\gamma = \tau^{\Phi^{-1}(\omega)}_\gamma \circ \Phi. \tag{2.26}$$

Proof Let γ be a path in M, and $\tilde{\gamma}_p$ be a horizontal lift starting at $p \in P$. Let us define $\zeta := \Phi \circ \tilde{\gamma}_p$. Then, denoting derivation with respect to t by an apostrophe,

$$\begin{aligned}(\Phi^{-1}(\omega))\left(\zeta'(t)\right) = ((\Phi^{-1})^*\omega)\left(\zeta'(t)\right) &= \omega\left((\Phi^{-1})_*[(\Phi \circ \tilde{\gamma}_p)'(t)]\right) \\ &= \omega\left((\Phi^{-1} \circ \Phi \circ \tilde{\gamma}_p)'(t)\right) = \omega\left(\tilde{\gamma}_p'(t)\right) = 0.\end{aligned}$$

Thus, ζ is a lift which is horizontal with respect to $\Phi^{-1}(\omega)$, whose starting point is $\zeta(0) = \Phi(\tilde{\gamma}_p(0)) = \Phi(p) = p\varphi(p)$. So the lift of γ which is horizontal in $\Phi^{-1}(\omega)$ and starting at p is

$$\tilde{\gamma}_p^{\Phi^{-1}(\omega)}(t) := \Phi(\tilde{\gamma}_p(t)\varphi(p)^{-1}),$$

and parallel transport becomes

$$\begin{aligned}\tau^{\Phi^{-1}(\omega)}_\gamma(p) = \tilde{\gamma}_p^{\Phi^{-1}(\omega)}(1) = \Phi(\tilde{\gamma}_p(1)\varphi(p)^{-1}) &= \Phi(\tau^\omega_\gamma(p)\varphi(p)^{-1}) \\ = \Phi(\tau^\omega_\gamma(p))\varphi(p)^{-1}.&\end{aligned}$$

Then

$$\tau^{\Phi^{-1}(\omega)}_\gamma(\Phi(p)) = \tau^{\Phi^{-1}(\omega)}_\gamma(p\varphi(p)) = \tau^{\Phi^{-1}(\omega)}_\gamma(p)\varphi(p) = \Phi(\tau^\omega_\gamma(p)). \qquad \square$$

Denote the isotropy group of a connection by $\Gamma_\omega := \{\Phi \in \mathscr{G} \mid \Phi(\omega) = \omega\}$.

Lemma 2.43 *Let Φ be a gauge transformation and ω a connection. The following statements are equivalent:*

- $\Phi \in \Gamma_\omega$.
- *$\phi_i(x)$ commutes with all elements of $\mathrm{Hol}^{\mathrm{L}}_x(\omega)$ for all x.*
- *$\varphi(p)$ commutes with all elements of $\mathrm{Hol}_p(\omega)$ for all p.*

Proof We have on the one hand

$$\psi_i(x, \phi_i(x)g^i_{x,\gamma}g_i) = \Phi(\psi_i(x, g^i_{x,\gamma}g_i)) = \Phi(\tau^\omega_\gamma(\psi_i(x, g_i)) = \Phi(\tau^\omega_\gamma(p))$$

and on the other

$$\psi_i(x, g^i_{x,\gamma}\phi_i(x)g_i) = \tau^\omega_\gamma(\psi_i(x, \phi_i(x)g_i)) = \tau^\omega_\gamma(\Phi(\psi_i(x, g_i))) = \tau^\omega_\gamma(\Phi(p)).$$

If $\Phi \in \Gamma_\omega$, then by Lemma 2.42 the right hand sides of these are equal so that the second statement of the proposition follows. On the other hand, if the second statement holds, then it follows that $\Phi(\tau^\omega_\gamma(p)) = \tau^\omega_\gamma(\Phi(p)) = \tau^{\Phi^{-1}(\omega)}_\gamma(\Phi(p))$, where the last equality follows from Lemma 2.42. Since parallel transport uniquely determines a connection, this implies $\Phi^{-1}(\omega) = \omega$, which in turn implies $\Phi \in \Gamma_\omega$.

The equivalence of the first point with the third point is proved in a similar fashion. Alternatively, it follows from the isomorphisms $\Gamma^\infty(M, \mathrm{Ad}\, P) \approx C^\infty_{\mathrm{Ad}}(P, G)$ of Theorem 1.18 and $\mathrm{Hol}^{\mathrm{L}}_x(\omega) \approx \mathrm{Hol}_p(\omega)$ of Proposition 2.39. $\square$

Proposition 2.44 *Denote $\tilde{\mathscr{G}} = \mathscr{G}/Z(\mathscr{G})$. Then $\tilde{\mathscr{G}}$ acts freely on the space of irreducible connections.*

Proof Suppose Φ does not act freely on ω, that is $\Phi \in \Gamma_\omega$. Then by the previous lemma, Φ commutes with elements of the holonomy of ω, which is all of G if ω is irreducible. So $\Phi \in Z(\mathscr{G})$. □

In the case of SU(2), for example, this means (recalling Proposition 1.19) $\Phi = \pm 1$.

Proposition 2.45 *Let ω be a connection, and let C_ω be the centralizer of $\mathrm{Hol}(\omega)$ in G, $C_\omega := \{g \in G \mid gh = hg \text{ for all } h \in \mathrm{Hol}(\omega)\}$. Then $\Gamma_\omega \cong C_\omega$.*

Proof We pick a certain trivialization (ψ, U) on which our isomorphism will depend. Take a $h \in C_\omega$. We can consider this h to be a constant function $h : U \to G$, and using equation (1.15) we can extend it piece by piece to all of M, to a family of functions $U_i \to G$ which are all constant. In this way, h determines an element of $\Gamma^\infty(M, \mathrm{Ad}\, P)$, i.e. a gauge transformation, which we will also denote by h. By assumption this gauge transformation satisfies the second point of Lemma 2.43, so that $h \in \Gamma_\omega$. This inclusion is obviously injective and homomorphic.

Conversely, let $\Phi \in \mathscr{G}$ be a gauge transformation. Then it induces a certain function $U \to G$, which we will also denote by h. Now assume h preserves a connection ω. Let A be the gauge potential of this connection on our trivialization U, so that[2] $h(A) = A$. Then if s' is another section then we may pull ω also back along s'. By Proposition 2.26 there is a function $\phi : U \to G$ such that $s'(x) = s(x)\phi(x)$, and the connection relative to this section takes the form $A' = \phi^{-1}A\phi + \phi^{-1}\mathrm{d}\phi$. Since A' is just another pullback of the connection ω, h must also leaves A' invariant, i.e.

$$\phi^{-1}A\phi + \phi^{-1}\mathrm{d}\phi = A' = h(A') = h^{-1}(\phi^{-1}A\phi + \phi^{-1}\mathrm{d}\phi)h + h^{-1}\mathrm{d}h,$$

or equivalently

$$\phi^{-1}A\phi + \phi^{-1}\mathrm{d}\phi - h^{-1}(\phi^{-1}A\phi + \phi^{-1}\mathrm{d}\phi)h = h^{-1}\mathrm{d}h.$$

This has to hold for all such functions ϕ. The left hand side depends on ϕ, while the right hand side does not, which can only be the case if the right hand side is in fact 0. This implies that h is constant; i.e. it is an element of G. By Lemma 2.43 it commutes with the holonomy of ω, so $h \in C_\omega$. □

2.2 Connections on vector bundles

We now explore connections on vector bundles. These are traditionally defined in a slightly different vocabulary (a differential map ∇ satisfying a certain Leibniz rule). However, we will soon see that just as with principal bundles, they may be described by certain local one-forms A, which induce a two-form F which is the square of ∇. Thus, such a map ∇ may be seen as the analog of the covariant derivative $\mathcal{D}_A$ on principal bundles.

Definition 2.46 Let $E \to M$ be a vector bundle of rank k. Then a *frame* is a set of sections $s_i : M \to E$ such that at each $x \in M$, $\{s_i(x)\}$ is a basis for E_x. They are in one-to-one correspondence with local trivializations.

Definition 2.47 Let $E \to M$ be a vector bundle. A *connection* on E is a linear map

$$\nabla : \Gamma(E) \to \Gamma(E \otimes \Omega^1 M) \tag{2.27}$$

[2] The reader might object here: by pulling back ω along different sections one may obtain different gauge potentials A that determine the same connection. Therefore, why should h leave A *invariant*, instead of mapping it to another A which is the same ω, but pulled back along another section? The answer to this is that a gauge transformation acting on a connection A along s is defined in such a way to be *the same as* pulling ω back along another section, namely the section $\Phi \circ s = sh$. From this it follows that h must leave A invariant.

satisfying the following Leibniz rule for any function f and section s:

$$\nabla(fs) = s \otimes \mathrm{d}f + f\nabla s. \tag{2.28}$$

A connection may thus be viewed as a map $\Omega^0(M, E) \to \Omega^1(M, E)$. Then there is a unique way to extend this to general E-valued forms on M:

$$\mathrm{d}_\nabla : \Omega^k(M, E) \to \Omega^{k+1}(M, E), \tag{2.29}$$

such that for $\omega \in \Omega^p(M)$, $\theta \in \Omega^q(M, E)$,

$$\begin{aligned} \mathrm{d}_\nabla &= \nabla \quad \text{on } \Omega^0(M, E), \\ \mathrm{d}_\nabla(\omega \wedge \theta) &= (\mathrm{d}\omega) \wedge \theta + (-1)^p \omega \wedge \mathrm{d}_\nabla \theta. \end{aligned}$$

Now let $\{e_i\}$ be a local frame for E.

Definition 2.48 The connection one-form A for a connection ∇ on a vector bundle is the unique matrix of one-forms defined by

$$\nabla e_i = \sum_{i=1}^{k} e_j \otimes A^j_i. \tag{2.30}$$

Let ω be a E-valued p-form, so we may write it as $\omega = \sum_i e_i \otimes \omega^i$ for forms $\omega^i \in \Omega^p(M)$. Then we calculate

$$\mathrm{d}_\nabla \omega = \sum_i \mathrm{d}_\nabla(e_i \otimes \omega^i) = \sum_i e_i \otimes \mathrm{d}\omega^i + \sum_{i,j} e_j \otimes A^j_i \wedge \omega^i = \mathrm{d}\omega + A \wedge \omega.$$

Here it is to be understood that d and A act on the components of s. This expression shows that a one-form A locally completely determines a connection, just as in the case of principal bundles. In fact, let us define $F = \mathrm{d}A + A \wedge A$.

Proposition 2.49 *We have*

$$\mathrm{d}^2_\nabla \omega = F \wedge \omega. \tag{2.31}$$

2.3 Connections on associated vector bundles

Let $P(M, G)$ be a principal bundle, V be a vector space, and $\rho : G \to \mathrm{End}(V)$ be a faithful linear representation. A connection on P then induces a connection on the associated vector bundle $P \times_\rho V$ in a natural way, which we now show. Since we primarily work in this thesis in terms of principal bundles, most of the proofs are omitted in this section.

Definition 2.50 Let $\alpha \in \Omega^k(P)$ be a form on P. If $h^*\alpha = \alpha$, then α is said to be *horizontal*. This extends to forms taking values in some vector space in the obvious way.

Definition 2.51 Let $\alpha \in \Omega^k(P, V)$ be a form on P taking values in the vector space V. Then it is said to be *basic* if it is horizontal and G-invariant, i.e.

$$R_g^* \alpha = \rho(g^{-1}) \circ \alpha.$$

The space of such forms is denoted by $\Omega^k_G(P, V)$. Their importance is because of the following.

Proposition 2.52 *Basic forms $\overline{\alpha} \in \Omega^k_G(P, V)$ are in one-to-one correspondence with forms on M taking values in the associated bundle, $\Omega^k(M, P \times_\rho V)$.*

Indeed, let s_i be the canonical sections associated to a set of trivializations of P. If $\overline{\alpha} \in \Omega^k_G(P, V)$ then $\alpha_i := s_i^*\overline{\alpha} \in \Omega^k(U_i, V)$ is a set of forms satisfying $\alpha_i = \rho(g_{ij}) \circ \alpha_j$, so that they define a global $P \times_\rho V$-valued form on M. Conversely, if we have a set of forms $\alpha_i \in \Omega^k(U_i, V)$ satisfying this condition on overlaps, then the set of $\overline{\alpha}_i := \rho(g_i^{-1}) \circ \pi^*\alpha_i \in \Omega^k(\pi^{-1}U_i, V)$ glues together to define a global form on P, which is basic. (Here g_i is the function defined by the trivialization as in Proposition 1.7).

Now, for basic forms we have a covariant exterior derivative D, by Definition 2.17.

Proposition 2.53 *Let $\overline{\alpha} \in \Omega^k_G(P, V)$. Then*

$$D\overline{\alpha} = \mathrm{d}\overline{\alpha} + \rho(\omega) \wedge \overline{\alpha}. \tag{2.32}$$

Here, ω is the connection one-form on P. ρ acts on it via the representation $\rho : \mathfrak{g} \to \mathrm{End}(V)$ associated with the Lie algebra-representation ρ. Also, $\wedge$ denotes both the wedge product on forms and the composition of the $\mathrm{End}(V)$-components of $\rho(\omega)$ with the V-components of $\overline{\alpha}$.

Proposition 2.54 *If $\overline{\alpha}$ is basic then $D\overline{\alpha}$ is basic as well.*

Proof $D\overline{\alpha} = h^*\mathrm{d}\overline{\alpha}$ is horizontal by construction. It is also G-invariant:

$$\begin{aligned}
R_g^*D\overline{\alpha} &= R_g^*h^*\mathrm{d}\overline{\alpha} \\
&= h^*R_g^*\mathrm{d}\overline{\alpha} && \text{(The Ehresmann-connection } H \text{ is equivariant)} \\
&= h^*\mathrm{d}R_g^*\overline{\alpha} \\
&= h^*\mathrm{d}(\rho(g^{-1}) \circ \overline{\alpha}) && (\overline{\alpha} \text{ is } G\text{-invariant}) \\
&= \rho(g^{-1}) \circ h^*\mathrm{d}\overline{\alpha} \\
&= \rho(g^{-1}) \circ D\overline{\alpha}.
\end{aligned}$$

□

Thus, let $\alpha \in \Omega^k(M, P \times_\rho V)$. Then it has an associated basic form $\overline{\alpha} \in \Omega^k_G(P, V)$. Since its covariant derivative $D\overline{\alpha} \in \Omega^{k+1}_G(P, V)$ is also basic, it comes from a form $\mathrm{d}_\omega\alpha \in \Omega^{k+1}(M, P \times_\rho V)$.

Proposition 2.55 *The map $\mathrm{d}_\omega : \Omega^k(M, P \times_\rho V) \to \Omega^{k+1}(M, P \times_\rho V)$ defined in this way is a connection on $P \times_\rho V$ in the sense of Definition 2.47.*

We have already seen how d_ω acts in terms of basic forms. We can also describe it in the following way.

Proposition 2.56 *Let $\alpha \in \Omega^k(M, P \times_\rho V)$ and let α_i be the corresponding forms $\alpha_i \in \Omega^k(U_i, V)$ satisfying $\alpha_i = \rho(g_{ij}) \circ \alpha_j$. Then $\mathrm{d}_\omega\alpha_i = \rho(g_{ij}) \circ \mathrm{d}_\omega\alpha_j$, so that $\mathrm{d}_\omega a_i$ defines a form in $\Omega^{k+1}(M, P \times_\rho V)$, and*

$$\mathrm{d}_\omega\alpha_i = \mathrm{d}\alpha_i + \rho(A_i) \wedge \alpha_j, \tag{2.33}$$

where A_i are the gauge potentials of the principal connection ω on P as in Definition 2.20.

Summarizing, suppose we have a principal bundle P, a representation ρ on a vector space V, and a connection. Then the connection on $P \times_\rho V$ is obtained by representing the connection of P on V via ρ. Depending on one's view, the connection one-form becomes $\rho(\omega)$ or $\rho(A_i)$.

Now we have seen that when one has a vector bundle there is also an associated principal G-bundle, called the *frame bundle*. Although we will not show it here, a connection on a vector bundle induces one on its frame bundle. These structures are each other inverses. Thus, since one may study a group either directly or via its representations, studying connections over principal G-bundles or vector bundles are essentially equivalent.

2.4 Connections on holomorphic vector bundles

Let M be a complex surface (i.e. it has complex dimension two, or real dimension four) and let $E \to M$ be a holomorphic vector bundle over it. Contrary to the smooth case, there exists a natural differential operator $\overline{\partial} : \Gamma(E) \to \Gamma(E \otimes \Omega^{0,1}(M))$. Here we shall study it and how it is related to connections as we already know them; it turns out that the study of holomorphic vector bundles is closely related to that of anti-self-dual connections (i.e. connections having an anti-self-dual field strength).

In Appendix B it is shown that there is the splitting $\Omega^2(M) = \Omega^2_+(M) \oplus \Omega^2_-(M)$ into self-dual and anti-self-dual two-forms.

Proposition 2.57 *If M is a Kähler surface with Kähler form ω, then we have*

$$\begin{aligned} \Omega^2_+(M) \otimes \mathbb{C} &= \Omega^{2,0}(M) \oplus \mathbb{C}\omega \oplus \Omega^{0,2}(M), \\ \Omega^2_-(M) \otimes \mathbb{C} &= \Omega^{1,1}(M) \cap \omega^{\perp}. \end{aligned} \tag{2.34}$$

Proof Choose a coordinate system x_1, x_2, x_3, x_4 with $z_1 = x_1 + ix_2$ and $z_2 = x_3 + ix_4$, such that the Kähler form becomes $\omega = \frac{i}{2}(\mathrm{d}z_1 \wedge \mathrm{d}\overline{z}_1 + \mathrm{d}z_2 \wedge \mathrm{d}\overline{z}_2)$. A direct computation shows that

$$\begin{aligned} \mathrm{d}\overline{z}_1 \wedge \mathrm{d}\overline{z}_2 &= (\mathrm{d}x_1 \wedge \mathrm{d}x_3 - \mathrm{d}x_2 \wedge \mathrm{d}x_4) + i(\mathrm{d}x_2 \wedge \mathrm{d}x_3 + \mathrm{d}x_1 \wedge \mathrm{d}y_4), \\ \mathrm{d}z_1 \wedge \mathrm{d}z_2 &= (\mathrm{d}x_1 \wedge \mathrm{d}x_3 - \mathrm{d}x_2 \wedge \mathrm{d}x_4) - i(\mathrm{d}x_2 \wedge \mathrm{d}x_3 + \mathrm{d}x_1 \wedge \mathrm{d}y_4), \\ \omega &= \mathrm{d}x_1 \wedge \mathrm{d}x_2 + \mathrm{d}x_3 \wedge \mathrm{d}x_4. \end{aligned}$$

The real and imaginary parts of the upper two lines, together with the bottom line, exactly span the space of self-dual forms. The statement follows. ☐

Definition 2.58 Denote by $\Omega^{p,q}(M, E)$ the space of (p, q)-forms that take values in E, take a holomorphic frame $\{e_i\}_i$ for E and write $\sigma = \sum_i \omega_i \otimes e_i$. We define the operator $\overline{\partial} : \Omega^{p,q}(M, E) \to \Omega^{p,q+1}(M, E)$ by

$$\overline{\partial}\sigma := \sum_i \overline{\partial}\omega_i \otimes e_i. \tag{2.35}$$

If $\{e'_i\}_i$ is another holomorphic frame of E, related to $\{e_i\}_i$ by $e_i = \sum_j g_{ij} e'_j$, then

$$\overline{\partial}\sigma = \overline{\partial} \sum_{i,j} g_{ij}\omega_i \otimes e'_j = \sum_{i,j} \overline{\partial}(g_{ij}\omega_i) \otimes e'_j = \sum_{i,j} g_{ij}\overline{\partial}\omega_i \otimes e'_j = \sum_i \overline{\partial}\omega_i \otimes e_i,$$

since the transition functions are holomorphic. Thus $\overline{\partial}$ is well defined.

Proposition 2.59 *The operator $\overline{\partial}$ satisfies*

- $\overline{\partial}(f\sigma) = \overline{\partial}f \otimes \sigma + f\overline{\partial}\sigma$ *for every smooth* $f : M \to \mathbb{C}$ *and* $\sigma \in \Gamma(E)$,
- $\overline{\partial}\sigma|_U = 0$ *if and only if σ is a holomorphic section on* E_U,
- $\overline{\partial}^2 = 0$.

This follows from straightforward computations. On the other hand, if E is a smooth bundle equipped with some connection ∇, then we have the operator $\mathrm{d}_\nabla : \Gamma(E) \to \Omega^1(M, E)$. The splitting $\Omega^1(M) \otimes \mathbb{C} = \Omega^{1,0}(M) \oplus \Omega^{0,1}(M)$ induces a corresponding splitting

$$\mathrm{d}_\nabla = \partial_\nabla + \overline{\partial}_\nabla, \tag{2.36}$$

with $\partial_\nabla : \Gamma(E) \to \Omega^{1,0}(M, E)$ and $\overline{\partial}_\nabla : \Gamma(E) \to \Omega^{0,1}(M, E)$.

Definition 2.60 Suppose that d_∇ is a connection splitting as $\mathrm{d}_\nabla = \partial_\nabla + \overline{\partial}_\nabla$. We say that d_∇ is *compatible with the complex structure* of E if $\overline{\partial}_\nabla = \overline{\partial}$.

Suppose that the bundle has a Hermitian metric, i.e. a fiberwise smooth metric h such that[3] $h_p(X,\overline{Y}) = \overline{h_p(Y,\overline{X})}$. Then we say that a connection ∇ is *compatible with the metric h* if $\mathrm{d}h(X,Y) = h(\mathrm{d}_\nabla X, Y) + h(X, \mathrm{d}_\nabla Y)$.

Proposition 2.61 *If E is a holomorphic vector bundle with a Hermitian metric, then there exists a unique connection which is compatible with both the complex structure and the metric of E.*

Proof Let $\{e_i\}_i$ be a holomorphic frame for E (i.e. the sections e_i are holomorphic). Write $h_{ij} = h(e_i, e_j)$. One may then write the connection as $d_\nabla e_j = \sum_i e_i A_{ij}$, where A_{ij} are one-forms. Since $0 = \overline{\partial} e_i = \overline{\partial}_\nabla e_i$, the one-forms A_{ij} must be of type (1,0) with respect to this holomorphic frame. Consequently (using the Einstein summation convention),

$$\begin{aligned}
\mathrm{d}h_{ij} &= \mathrm{d}(e_i, e_j) = h(\mathrm{d}_\nabla e_i, e_j) + h(e_i, \mathrm{d}_\nabla e_j) \\
&= h(e_k A_{ki}, e_j) + h(e_i, e_k A_{kj}) = h_{kj} A_{ki} + h_{ik} \overline{A}_{kj} \\
&= A^\top h + h\overline{A}.
\end{aligned}$$

The first term of this is of type (1,0) and the second of type (0,1). On the other hand, we have $\mathrm{d}h_{ij} = \partial h_{ij} + \overline{\partial} h_{ij}$, of which the first term is again of type (1,0) and the second of type (0,1). This implies

$$\partial h = A^\top h \quad \text{and} \quad \overline{\partial} h = h\overline{A}.$$

Both of these are solved only if $A = h^{-1}\partial h$. Since A is determined by the conditions of compatibility, A is well-defined globally. $\square$

The operator $\overline{\partial}_\nabla$ resembles the operator $\overline{\partial}$ that we have seen above. We have already seen that there are connections d_∇ such that $\overline{\partial} = \overline{\partial}_\nabla$. On the other hand, suppose we have a connection of which the (0,1) part is $\overline{\partial}_\nabla$. A natural question then is whether there exist holomorphic structures on E such that its associated $\overline{\partial}$ operator equals $\overline{\partial}_\nabla$. At the very least, then, $\overline{\partial}_\nabla$ has to square to zero, because of the last point of Proposition 2.59. In fact, as a consequence of the following general theorem we will see that this is enough.

Theorem 2.62 *Let $E \to M$ be a smooth complex fiber bundle over the complex manifold M. If $\overline{\partial} : \Omega^0(E) \to \Omega^{0,1}(E)$ is an operator such that*

- *$\overline{\partial}(f\sigma) = \overline{\partial} f \otimes \sigma + f\overline{\partial}\sigma$ for every smooth $f : M \to \mathbb{C}$ and $\sigma \in \Gamma(E)$,,*
- *its natural extension to forms of arbitrary degrees satisfies $\overline{\partial}^2 = 0$,*

then it defines a holomorphic structure on E. Two such operators give isomorphic holomorphic structures if and only if they are conjugate by an automorphism of the bundle.

Corollary 2.63 *A connection d_∇ on a smooth complex bundle E over a complex surface M defines a holomorphic structure on E if and only if $\overline{\partial}_\nabla^2 = 0$. In this case the operator of Definition 2.58 takes the form $\overline{\partial} = \overline{\partial}_\nabla$.*

The operators $\overline{\partial}$ and $\overline{\partial}_\nabla$ are called *Cauchy-Riemann operators*. For a proof, see [9].

We may decompose F_∇ as $F_\nabla = F_\nabla^{2,0} + F_\nabla^{1,1} + F_\nabla^{0,2}$.

Corollary 2.64 *A connection ∇ on a smooth bundle E defines a holomorphic structure on E if and only if $F_\nabla^{0,2} = 0$.*

If the bundle is Hermitian, then the structure group can be reduced to $\mathrm{U}(n)$. If the connection is compatible with the metric, then the Lie algebra-components of A and F must be in $\mathfrak{u}(n)$. This implies that $A_{ij} = -\overline{A}_{ji}$, and $F_\nabla^{2,0} = -\overline{F_\nabla^{0,2}}$. Thus in this case, F_∇ must be a $(1,1)$-form. Comparing with Proposition 2.57 then gives the following two results.

[3] In the case $E = TM$, for example, if g is a metric compatible with the complex structure J, then such a metric is given by $h = g + i\omega$. In fact, this is the reason why we call M Hermitian: a Riemannian metric compatible with J induces a metric $h = g + i\omega$ which is smooth and Hermitian on the fibers. One could also start with this definition, and then retrieve g from h by $g = \mathrm{Re}\, h = \frac{1}{2}(h + \overline{h})$, and $\omega = \mathrm{Im}\, h = \frac{i}{2}(h - \overline{h})$.

Corollary 2.65 *If E is a smooth vector bundle over a Kähler surface M, then any anti-self-dual $U(n)$-connection on E defines a holomorphic structure on E. Conversely, if E is a holomorphic vector bundle over a Kähler surface M, then a $U(n)$-connection ∇ compatible with E is anti-self-dual if and only if*

$$F_\nabla \perp \omega. \tag{2.37}$$

We shall return to this subject in section 5.2, where we consider its impact on the moduli spaces of anti-self-dual connections.

2.5 Characteristic classes

Definition 2.66 Suppose P is a totally symmetric n-linear polynomial on a Lie algebra $\mathfrak{g}$. Then we say it is a *symmetric invariant polynomial* if

$$P(g^{-1}X_1g, \ldots, g^{-1}X_ng) = P(X_1, \ldots, X_n). \tag{2.38}$$

An *invariant polynomial of degree n* is defined as a symmetric invariant polynomial with all its entries equal:

$$P_n(X) := P(\underbrace{X, \ldots, X}_{n}) =: P(X^n). \tag{2.39}$$

Definition 2.67 Now let ω be a $\mathfrak{g}$-valued p-form, $\omega = \alpha \otimes X$, where α is a p-form and X an element of $\mathfrak{g}$. Then if P is a symmetric invariant polynomial, we extend the previous definition as

$$P(\omega_1, \ldots, \omega_n) = P(X_1, \ldots, X_n)\alpha_1 \wedge \cdots \wedge \alpha_n. \tag{2.40}$$

The diagonal combination is again called an invariant polynomial of degree n:

$$P_n(\omega) := P(\underbrace{\omega, \ldots, \omega}_{n}). \tag{2.41}$$

Lemma 2.68 *Denoting by p_i the degree of ω_i, we have*

$$\mathrm{d}P(\omega_1, \ldots, \omega_n) = \sum_{i=1}^{n} (-1)^{p_1+\cdots+p_{i-1}} P(\omega_1, \ldots, \mathcal{D}\omega_i, \ldots, \omega_n). \tag{2.42}$$

Now let $E \to M$ be a vector bundle, having a connection ∇ with curvature $F = \nabla^2$. Alternatively, if we have a principal bundle with a connection on it, then we may represent it on an associated vector bundle to gain a curvature F.

Theorem 2.69 **Chern-Weil.** *Let $P_n(F)$ be an invariant polynomial, and F the curvature two-form of a connection. Then*

- *$P_n(F)$ is closed; $\mathrm{d}P_n(F) = 0$.*
- *If F and F' are two curvature 2-forms corresponding to two different connections on the same bundle, the difference $P_n(F) - P_n(F')$ is exact.*

The first item follows from Lemma 2.68. It implies that $P_n(F)$ is a representative of a certain cohomology class. The second item says that all such polynomials $F_n(F)$ are in the same cohomology class, regardless of F; in other words, the class $[P_n(F)]$ is independent of the connection.

Theorem 2.70 **Naturality of characteristic classes.** *Let $f : N \to M$ be a differentiable map, and $E \to M$ a vector bundle over M. Suppose $P_n(F)$ is an invariant polynomial, and denote its characteristic class by $[P_n(E)]$. Then*

$$[P_n(f^*E)] = f^*[P_n(E)]. \tag{2.43}$$

Since $P_n(F)$ is closed, it can locally be written as the d of some form:

$$P_n(F) = \mathrm{d}Q_{2n-1}(A, F), \tag{2.44}$$

where $Q_{2n-1}(A, F) \in \mathfrak{g} \otimes \Omega^{2n-1}(M)$. This form is called the *Chern-Simons form* of $P_n(F)$, and it is given by

$$Q_{2n-1}(A, F) = n \int_0^1 \mathrm{d}t\, P(A, F_t, \ldots, F_t), \tag{2.45}$$

where $F_t = t\mathrm{d}A + t^2 A \wedge A$.

Proposition 2.71 **Splitting principle.** *Let $E \to M$ be a vector bundle of rank k over M. Then there exists a space M' and a continuous map $f : M' \to M$ such that*

- *The induced cohomology homomorphism $f^* : H(M) \to H(M')$ is injective,*
- *The pullback bundle f^*E over M' splits into a direct sum of line bundles,*

$$f^*E = L_1 \oplus \cdots \oplus L_k, \tag{2.46}$$

for certain line bundles L_i.

2.5.1 Chern classes

Definition 2.72 Let $E \to M$ be a complex vector bundle of rank k, with a connection on it. The *total Chern class* is defined by

$$c(F) = \det\left(1 + \frac{i}{2\pi} F\right). \tag{2.47}$$

Since F is a two-form, $c(F)$ is a sum of forms of even degrees:

$$c(F) = 1 + c_1(F) + c_2(F) + \cdots \tag{2.48}$$

The forms $c_n(F) \in \Omega^{2n}(M)$ are called the *n-th Chern classes*.

By diagonalizing F and expanding the determinant, it may be seen that these Chern classes take the form

$$\begin{aligned} c_1(F) &= \frac{i}{2\pi}\,\mathrm{Tr}(F), \\ c_2(F) &= \frac{1}{8\pi^2}[\mathrm{Tr}(F \wedge F) - \mathrm{Tr}(F) \wedge \mathrm{Tr}(F)], \\ &\vdots \\ c_k(F) &= \left(\frac{i}{2\pi}\right)^{2k} \det(F). \end{aligned} \tag{2.49}$$

They are invariant polynomials. Hence the cohomology classes $c_i(E) := [c_i(F)]$ and the numbers

$$c_n := \int_M c_n(E), \tag{2.50}$$

called *Chern numbers*, are independent of the connection. Moreover, if M is compact, then the Chern numbers are integers.

If M is an almost complex manifold (see Appendix A), then it has a complex tangent bundle. In this case we write $c_i(M) := c_i(TM)$.

Proposition 2.73 *Let E be a complex vector bundle of rank k.*

- $c_i(E) = 0$ *if* $i > k$ *or* $2i > \dim M$.
- *The Whitney sum formula: if E' is another complex vector bundle, then*

$$c(E \oplus E') = c(E) \wedge c(E'). \tag{2.51}$$

  To be specific, $c_i(E \oplus F) = \sum_{i=j+k} c_j(E) \wedge c_k(F)$.
- *If E is trivial, then* $c(E) = 1$.

Proof The first item is trivial. For the second item, suppose ∇_E and $\nabla_{E'}$ are connections on E and E', respectively. Then there is a natural direct sum connection $\nabla_E \oplus \nabla_{E'}$ on $E \oplus E'$, having connection one-forms $\mathrm{diag}(A_E, A_{E'})$. The curvature then is $F_{E\oplus E'} = \mathrm{diag}(F_E, F_{E'})$, and

$$\begin{aligned} c(E \oplus E') &= \det\left(1 + \frac{i}{2\pi} F_{E\oplus E'}\right) = \det\left(1 + \frac{i}{2\pi} F_E\right) \wedge \det\left(1 + \frac{i}{2\pi} F_{E'}\right) \\ &= c(E) \wedge c(E'). \end{aligned}$$

As for the third item, let $E = M \times \mathbb{C}^k = M \times \left(\bigoplus_{i=1}^k \mathbb{C}\right)$. On the trivial bundle $M \times \mathbb{C}$, the ordinary exterior derivative d is a connection, which squares to 0 so that $F_{\mathrm{d}} = 0$. Therefore, by the second item, $c(E) = \bigwedge_{i=1}^k c(\mathbb{C}) = 1$. □

Let E be a complex bundle and f be a map as in Proposition 2.71, i.e. $f^*E = L_1 \oplus \dots L_k$. Then

$$c(E) = \prod_i c(L_i) = \prod_i (1 + c_1(L_i)) = \prod_i (1 + x_i), \tag{2.52}$$

where $x_i := c_1(L_i)$. In particular,

$$\begin{aligned} c_1(E) &= \sum_i x_i, \\ c_2(E) &= \sum_{i_1 < i_2} x_{i_1} x_{i_2}, \\ &\vdots \\ c_j(E) &= \sum_{i_1 < \dots < i_j} x_{i_1} \dots x_{i_j}. \end{aligned} \tag{2.53}$$

2.5.2 Chern characters

Definition 2.74 Let $E \to M$ again be a complex vector bundle of rank k. Then we define its total *Chern character* by

$$\mathrm{ch}(E) := \mathrm{Tr} \exp\left(\frac{iF}{2\pi}\right) = \sum_j \frac{1}{j!} \mathrm{Tr}\left(\left(\frac{iF}{2\pi}\right)^j\right). \tag{2.54}$$

In particular, if L is a complex line bundle then $\mathrm{ch}(L) = e^{c_1(L)}$. They are important to us because of this relationship with the Chern classes, and because they behave well under direct sums and tensor products of bundles.

Theorem 2.75 *If E and E' are two complex vector bundles, then*

$$\begin{aligned} \mathrm{ch}(E \oplus E') &= \mathrm{ch}(E) + \mathrm{ch}(E'), \\ \mathrm{ch}(E \otimes E') &= \mathrm{ch}(E) \wedge \mathrm{ch}(E'). \end{aligned} \tag{2.55}$$

Proof For the direct sum, one can use the connection shown in the proof of Theorem 2.73 and the properties of the trace to prove the assertion. For the tensor product, let ∇_E and $\nabla_{E'}$ be connections on E and E' respectively. Then we may define a connection $\nabla_{E\otimes E'}$ on $E \otimes E'$ to be the unique connection satisfying

$$\nabla_{E\otimes E'}(s \otimes s') = \nabla_E(s) \otimes s' + s \otimes \nabla_{E'}(s')$$

on pure tensors. Then $F_{E\otimes E'} = F_E \otimes I + I \otimes F_{E'}$, and the assertion follows again from basic properties of the trace. □

Proposition 2.76 *Let L and L' be two complex line bundles. Then their tensor product, which is again a line bundle, has as first Chern class*

$$c_1(L \otimes L') = c_1(L) + c_1(L'). \tag{2.56}$$

Proof We have on the one hand

$$\mathrm{ch}(L \otimes L') = e^{c_1(L\otimes L)} = 1 + c_1(L \otimes L)$$

but on the other hand

$$\begin{aligned}\mathrm{ch}(L \otimes L') &= \mathrm{ch}(L) \wedge \mathrm{ch}(L') = e^{c_1(L)} \wedge e^{c_1(L')} \\ &= (1 + c_1(L)) \wedge (1 + c_1(L')) = 1 + c_1(L) + c_1(L').\end{aligned}$$

□

Proposition 2.77 *If E is a complex vector bundle of rank k, then*

$$c_1(\det E) = c_1(E). \tag{2.57}$$

Proof Suppose E splits into line bundles $f^*E = L_1 \oplus \cdots \oplus L_k$ via Proposition 2.71. Then it is easy to see that $\det E = \bigwedge^k E$ splits as $L_1 \otimes \cdots \otimes L_k$. By the previous proposition, this has first Chern class $c_1(L_1) + \cdots + c_1(L_k)$ which is just that of E. □

Proposition 2.78 *If E^* is the vector bundle dual to E, then*

$$c_i(E^*) = (-1)^i c_i(E). \tag{2.58}$$

Proof If L is an arbitrary line number, then $0 = c_1(M \times \mathbb{C}) = c_1(L \otimes L^*) = c_1(L) + c_1(L^*)$, i.e. $c_1(L^*) = -c_1(L)$. Therefore, by the splitting principle,

$$c(E) = c(L_1^*) \dots c(L_k^*) = \prod_i (1 + c_1(L_i^*)) = \prod_i (1 - c_1(L_i))$$

The result then follows from equations (2.53). □

Corollary 2.79 *If a complex vector bundle E is isomorphic to its dual E^*, then the odd Chern classes vanish.*

2.5.3 Pontryagin classes

Definition 2.80 Let $E \to M$ be a real vector bundle of rank k. Then we define the i-th *Pontryagin class* to be

$$p_i(E) = (-1)^i c_{2i}(E \otimes \mathbb{C}) \in H^{4i}(M, \mathbb{Z}). \tag{2.59}$$

As usual, there is a total Pontryagin class $p(E) = \sum_i p_i(E)$. Alternatively, one may show that this is equivalent to $p(E) = \left[\det\left(1 + \frac{1}{2\pi}F\right)\right]$.

If M is a smooth manifold then we write $p_i(M) := p_i(TM)$.

Just as in the case of complex vector spaces, one can use a Hermitian metric on a complex bundle E to see that $E^* \cong \overline{E}$. Here E^* is the dual of E and $\overline{E}$ is the complex bundle consisting of the conjugated fibers, $\overline{E}_p = \overline{E_p}$.

Lemma 2.81 *Let $E \to M$ be a complex vector bundle of rank k. Denote by $E_{\mathbb{R}}$ the same vector bundle considered as a real vector bundle of rank $2k$. Then $E_{\mathbb{R}} \otimes \mathbb{C} \cong E \oplus E^*$.*

Proof Let $u + v \otimes i \in (E_{\mathbb{R}} \otimes \mathbb{C})_p$ and consider the map

$$\phi(u + v \otimes i) = (u + iv, \overline{u - iv}) \in E_p \oplus \overline{E}_p \cong E_p \oplus E_p^*.$$

This is clearly linear over $\mathbb{R}$. Moreover,

$$\begin{aligned} \phi(i(u + v \otimes i)) = \phi(-v + u \otimes i) &= (-v + iu, \overline{-v - iu}) = (i(u + iv), \overline{-i(u - iv)}) \\ &= i(u + iv, \overline{u - iv}) = i\phi(u + v \otimes i). \end{aligned}$$

Thus it is linear over $\mathbb{C}$. □

Corollary 2.82 *Writing p_i for $p_i(E)$ and c_i for $c_i(E)$, we have*

$$1 - p_1 + p_2 + \cdots + (-1)^k p_k = c(E) \wedge (1 - c_1 + c_2 + \cdots + (-1)^n c_n). \tag{2.60}$$

Proof On the one hand $c(E_{\mathbb{R}} \otimes \mathbb{C}) = \sum_i c_i(E_{\mathbb{R}} \otimes \mathbb{C})$. But $E_{\mathbb{R}}$ is isomorphic to its dual $E_{\mathbb{R}}^*$, so $E_{\mathbb{R}} \otimes \mathbb{C}$ is isomorphic to its dual, so that the odd Chern classes vanish from this sum, while the even ones satisfy $c_{2i}(E_{\mathbb{R}} \otimes \mathbb{C}) = (-1)^i p_i(E_{\mathbb{R}})$. Therefore, $c(E_{\mathbb{R}} \otimes \mathbb{C}) = \sum_i (-1)^i p_i(E_{\mathbb{R}})$.

On the other hand, by the Whitney sum formula (Proposition 2.73): $c(E_{\mathbb{R}} \otimes \mathbb{C}) = c(E \oplus \overline{E}) = c(E) \wedge c(\overline{E}) = c(E) \wedge \sum_i (-1)^i c_i(E)$. □

In particular, one has

$$p_1(E) = c_1(E)^2 - 2c_2(E). \tag{2.61}$$

2.5.4 The Euler class

Definition 2.83 Let $E \to M$ be a real oriented vector bundle of even rank k. Since the curvature two-form F is a skew-symmetric matrix, one can consider its Pfaffian[4]. We define the *Euler class* by

$$e(E) = \left[\frac{1}{(2\pi)^{k/2}} \operatorname{Pf}(F) \right]. \tag{2.62}$$

Upon comparison with the top Pontryagin class we immediately see that $e(E)^2 = p_n(E)$.

Using similar arguments as in the proof of the Whitney sum formula for the Chern class, one can show the following.

Proposition 2.84 *The Euler class also satisfies the Whitney sum formula, i.e.*

$$e(E \oplus F) = e(E) \wedge e(F). \tag{2.63}$$

Proposition 2.85 *If $E \to M$ is a complex vector bundle of rank k then $e(E) = c_k(E)$.*

Proof Consider the case when E is a line bundle. The connection two-form F then is a 1×1 complex matrix, or if we view E as a rank-two real bundle, a 2×2 real matrix. A moment's thought then shows that $c_1(E) = e(E)$. The general case then follows from the splitting principle and the Whitney sum formulas for c and e. □

As always, we write $e(M) := e(TM)$.

Theorem 2.86 **Gauss-Bonnet.** *Let M be an an oriented closed compact manifold of real dimension $2n$. Then*

$$\chi(M) = \int_M e(M). \tag{2.64}$$

This theorem will be of particular importance to us later on. We shall sketch a proof of it in section 6.3.

[4]For a brief introduction of the Pfaffian of a skew-symmetric matrix, see Appendix C.3.

2.5.5 The signature of a manifold

Let M be a compact oriented manifold of even dimension $2n$.

Definition 2.87 Consider the symmetric bilinear form $Q : H^n(M,\mathbb{R}) \times H^n(M,\mathbb{R}) \to H^{2n}(M,\mathbb{R}) \cong \mathbb{R}$ given by the wedge product. This form is nondegenerate, and we call it the *intersection form*. Alternatively, if $\smile$ denotes the cup product $\smile : H^p(M,\mathbb{Z}) \times H^q(M,\mathbb{Z}) \to H^{p+q}(M,\mathbb{Z})$, then Q may be evaluated by $Q(a,b) = \langle a \smile b, [M]\rangle$ for $a, b \in H^n(M,\mathbb{Z})$. In this case, Q takes values in $\mathbb{Z}$. When $Q(a,a)$ is even for all $a \in H^p(M,\mathbb{Z})$ then we say that Q is *even*.

Let b^+ and b^- be the number of positive and negative eigenvalues, respectively, of Q.

Definition 2.88 If M is a manifold of dimension $4n$, $n \in \mathbb{N}$, then we define its *signature* as $\tau(M) := b^+ - b^-$.

Proposition 2.89 *The signature has the following properties.*

- *If $\overline{M}$ is M with the reversed orientation, then $\tau(\overline{M}) = -\tau(M)$.*
- *It is multiplicative, i.e. $\tau(M \times N) = \tau(M)\tau(N)$.*
- *It is additive under disjoint unions, $\tau(M \sqcup N) = \tau(M) + \tau(N)$.*
- *If M is 4-dimensional, then*

$$\tau(M) = \frac{1}{3}p_1, \tag{2.65}$$

where p_1 is the Pontryagin number $p_1 := \int_m p_1(M)$.

2.5.6 Stiefel-Whitney classes

Besides SU(2)-bundles, we shall also encounter bundles with $\mathrm{SO}(n)$ as structure group. This group is not simply connected; however, it has a universal double cover $\mathrm{Spin}(n)$ which is simply connected. (In the case $n = 3$, there is the isomorphism $\mathrm{Spin}(3) = \mathrm{SU}(2)$.) An important question is whether an $\mathrm{SO}(n)$-bundle can be lifted to a Spin-bundle. The answer to this question will be given by the second *Stiefel-Whitney* class, while we can use the first to determine of a manifold is orientable or not. They are defined in terms of Čech cohomology over (in the case of $\mathrm{O}(n)$-bundles) $\mathbb{Z}_2$. For a brief introduction on this cohomology for this particular coefficient group, see section C.4 in the appendix.

Let $E \to M$ be a vector bundle of rank n with structure group $\mathrm{O}(n)$. Let $\mathcal{U} = \{U_i\}_i$ be a simple covering of M, i.e. one for which the intersection of any number of charts is either empty or contractible. Let $\{e_{i\alpha}\}$, $1 \leq \alpha \leq \mathrm{rank}(E)$ be an orthonormal frame, so that $e_{i\alpha} = t_{ij}e_{j\alpha}$ on overlaps. We define an element $f \in \check{C}^1(\mathcal{U},\mathbb{Z}_2)$ by

$$f_{ij}(E) = \det(g_{ij}) = \pm 1, \tag{2.66}$$

where $g_{ij} : U_i \cap U_j \to \mathrm{O}(n)$ are the transition functions. By the second equality of proposition 1.1 it is symmetric in its indices, and by the third it is closed under δ, so that it indeed an element $[f]$ of the cohomology.

Definition 2.90 We define the *first Stiefel-Whitney class* as $w_1(E) = [f] \in \check{H}(M,\mathbb{Z}_2)$.

Proposition 2.91 *This class does not depend on the orthonormal frame and its corresponding transition functions.*

Proof Let $\bar{e}_{i\alpha}$ be another orthonormal frame, related to $e_{i\alpha}$ by a matrix $h_i \in \mathrm{O}(n)$. Then $\overline{g}_{ij} = h_i g_{ij} h_j^{-1}$. Defining the 0-cochain $f_0(i) = \det h_i$, we find

$$\overline{f}(i,j) = \det(h_i g_{ij} h_j^{-1}) = \det(h_i)\det(h_j)\det(g_{ij}) = (\delta f_0)(i,j)f(i,j),$$

so that $[\overline{f}] = [f] = w_1(E)$. (Recall that we are using multiplicative notation.) □

Theorem 2.92 *A Riemannian manifold M is orientable if and only if $w_1(M) := w_1(TM)$ is trivial.*

Proof If M is orientable then the structure group $\mathrm{O}(n)$ may be reduced to $\mathrm{SO}(n)$, so that $f(i,j) = \det(g_{ij}) = 1$ for any i and j, and therefore $w_1(E) = 1$.

Conversely, suppose that $w_1(TM)$ is trivial, i.e. $f = \delta f_0$ for some f_0. Thus $f(i,j) = f_0(j)f_0(i)$. Choose matrices h_i such that $\det(h_i) = f_0(i)$, and consider the frame $\bar{e}_i = h_i e_i$. This is then an oriented frame. Indeed,

$$\begin{aligned}\det(\bar{g}_{ij}) &= \det(h_i)\det(h_j)\det(g_{ij}) = \det(h_i)\det(h_j) f(i,j)\\ &= f_0(i)f_0(j)f_0(j)f_0(i) = 1.\end{aligned}$$

□

Now, suppose that the structure group is $\mathrm{SO}(n)$, and consider a lifting of the transition functions $\tilde{g}_{ij} : U_i \cap U_j \to \mathrm{Spin}(n)$ such that $\varphi(\tilde{g}_{ij}) = g_{ij}$ and $\tilde{g}_{ij} = \tilde{g}_{ji}^{-1}$, where φ is the $2:1$ homomorphism $\mathrm{Spin}(n) \to \mathrm{SO}(n)$. Such a lifting always exists locally. Since

$$\varphi(\tilde{g}_{ij}\tilde{g}_{jk}\tilde{g}_{ki}) = g_{ij}g_{jk}g_{ki} = I, \tag{2.67}$$

we have $\tilde{g}_{ij}\tilde{g}_{jk}\tilde{g}_{ki} = \pm 1$. We define the Čech 2-cochain f by

$$f(i,j,k)I = \tilde{g}_{ij}\tilde{g}_{jk}\tilde{g}_{ki}. \tag{2.68}$$

Definition 2.93 We define the *second Stiefel-Whitney-class* as $w_2(E) = [f] \in \check{H}^2(M, \mathbb{Z}_2)$.

By using similar arguments as for $w_1(E)$, it is easy to see that f is symmetric, closed and independent of the chosen frame.

Theorem 2.94 *An* $\mathrm{SO}(n)$*-bundle* E *can be lifted to a* $\mathrm{Spin}(n)$*-bundle if and only if* $w_2(E)$ *is trivial.*

Proof If E is spin then by definition there are transition functions $\tilde{g}_{ij}$ such that $f(i,j,k)I = \tilde{g}_{ij}\tilde{g}_{jk}\tilde{g}_{ki} = I$, so that $w_2(E) = 1$.

On the other hand, suppose that $w_2(E)$ is trivial, i.e. there is a 1-cochain f_1 such that

$$f(i,j,k) = \delta f_1(i,j,k) = f_1(j,k)f_1(i,k)f_1(i,j).$$

If we choose new transition functions $\tilde{g}'_{ij} = \tilde{g}_{ij} f_1(i,j)$, then

$$\begin{aligned}\tilde{g}'_{ij}\tilde{g}'_{jk}\tilde{g}'_{ki} &= \tilde{g}_{ij}\tilde{g}_{jk}\tilde{g}_{ki} f_1(i,j)f_1(j,k)f_1(k,i)\\ &= f(i,j,k)I\delta f_1(i,j,k) = f(i,j,k)^2 I = I,\end{aligned}$$

so that the $\tilde{g}'_{ij}$ form the transition functions of a Spin-bundle. □

If $\check{H}^1(M, \mathbb{Z}_2)$ is trivial then this spin-structure is unique.

Corollary 2.95 *An orientable Riemannian smooth manifold is spin if and only if* $w_2(M)$ *is trivial.*

Like the other classes, these first two Stiefel-Whitney classes are two particular terms of a general Stiefel-Whitney class $w(E) \in H^*(M, \mathbb{Z}_2)$. Instead of providing the full definition of $w(E)$, which is rather involved, we mention only that it is uniquely determined by the following axioms:

- Like the other classes it is natural, i.e. for a map $f : M' \to M$ we have $w(f^*E) = f^*w(E)$.
- $w_0(E) = 1 \in H^0(M, \mathbb{Z}_2)$.
- $w_1(\gamma^1)$ is the generator of $H^1(\mathbb{RP}^1, \mathbb{Z}_2) = \mathbb{Z}_2$, where γ^1 is the tautological line bundle (see section 3.1.2).
- Whitney sum formula: $w(E \oplus F) = w(E) \smile w(F)$.

We now list a number of results which we need later on, but which we cannot prove because the proofs would be too involved. For a more comprehensive treatment of these subjects, see for example [10].

Proposition 2.96 *If $E \to M$ is a rank k complex bundle, then one has $c_i(E) \equiv w_{2i}(E) \bmod 2$ for all $i \leq k$. If M is an n-dimensional smooth manifold then we have $e(M) \equiv w_n(M) \bmod 2$.*

In particular, then, a complex surface M is spin if and only if its first Chern class satisfies $c_1(M) \equiv 0 \bmod 2$.

The following is a specialization of Wu's formula. It is most easily seen when $H^1(M, \mathbb{Z})$ has no torsion (for example when M is simply connected); then via the universal coefficient theorem (Theorem C.10) any mod 2 class is the reduction of an integral class modulo 2, so that it can be represented by an oriented embedded surface Σ. One has

$$\begin{aligned}\langle w_2(TM), [\Sigma]\rangle &= \langle w_2(T\Sigma \oplus N_\Sigma), [\Sigma]\rangle \\ &= \langle w_2(T\Sigma), [\Sigma]\rangle + \langle w_2(N_\Sigma), [\Sigma]\rangle + \langle w_1(T\Sigma) w_1(N_\Sigma), [\Sigma]\rangle,\end{aligned}$$

where N_Σ is the normal bundle. Now, w_2 is the mod 2 reduction of the Euler class in the case of surfaces. Then the first term drops out because the Euler characteristic $\langle e(T\Sigma), [\Sigma]\rangle = \chi(\Sigma) = 2 - 2g(\Sigma)$ is even. The last term drops out because Σ is orientable, while the second becomes the mod 2 reduction of $e(N_\Sigma) = \Sigma.\Sigma$, the self-intersection number. This gives the following.

Proposition 2.97 *If M is a four-dimensional smooth manifold, and $a \in H^2(M, \mathbb{Z}_2)$, then*

$$Q(w_2(M), a) \equiv Q(a, a) \bmod 2. \tag{2.69}$$

Corollary 2.98 *If M is spin then its intersection form is even. If $H^1(M, \mathbb{Z})$ has no torsion, then the converse also holds.*

These invariants cannot be arbitrarily chosen, however. They are related through the *Pontryagin square* of w_2, which, although defined more generally, is particularly easy to describe when M has no torsion in $H^1(M, \mathbb{Z})$.

Definition 2.99 Let M be a four-manifold. Let $v \in H^2(M, \mathbb{Z}_2)$, and pick a lift of v to $H^2(M, \mathbb{Z}_2)$ via the universal coefficient theorem. Denoting this lift by v', we set

$$v^2 := Q(v', v') \bmod 4. \tag{2.70}$$

That this is independent of the lift v' is immediate: suppose $v'' = v' + 2w$ is another lift of v. Then

$$Q(v'', v'') = Q(v' + 2w, v' + 2w) = Q(v', v') + 4(Q(v', w) + Q(w, w)).$$

Theorem 2.100 *If M is four-dimensional, then the isomorphism class of a bundle E is completely determined by $w_2(E)$ and $c_2(E)$. Moreover,*

$$w_2(E)^2 = p_1(E) \bmod 4, \tag{2.71}$$

and any $w_2 \in H^2(M, \mathbb{Z}_2)$ and p_1 related in this way uniquely determine an SO(3)*-bundle ([11]).*

Corollary 2.101 *If E is an* SO(3) *bundle over a spin four-manifold, then $p_1(E)$ is even.*

Proof Lift $w_2(E)$ to some $v \in H^2(M, \mathbb{Z})$. Then

$$p_1(E) \bmod 2 = w_2(E)^2 \bmod 2 = Q(v, v) \bmod 2 = 0. \qquad \square$$

Chapter 3

K3 surfaces

This chapter will introduce and explore K3 surfaces, which we will take as the base space of our principal bundles in the rest of this thesis. All of the proofs in the first section can be found in [7, 12–14]. In this chapter and in fact the remainder of this thesis, the word 'surface' means a *complex* surface: a complex manifold of complex manifold 2. In particular, such a complex surface is a smooth manifold of (real) dimension 4.

3.1 Tori and projective space

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function such that $0 \in \mathbb{C}$ is a regular value. Let $M := f^{-1}(0) \subset \mathbb{C}^n$. Then by the implicit function theorem, M is a complex manifold of dimension $n-1$.

3.1.1 Complex tori

Let Λ be a full lattice in $\mathbb{C}^n$, i.e. a discrete subgroup of $\mathbb{R}^{2n}$ spanning $\mathbb{R}^{2n}$. Let $M = \mathbb{C}^n/\Lambda$ endowed with the quotient topology. If $U \subset \mathbb{C}^n$ is smaller than a cell, i.e. $(U+\lambda) \cap U = \emptyset$ for all $\lambda \in \Lambda$, then $U \to \pi(U)$ is bijective (here π is the projection $\mathbb{C}^n \to M$). Covering M by these provides a holomorphic atlas. For example, if $\Lambda = \mathbb{Z}^{2n}$ then the polydisc $U = B_\epsilon(z)$ with $\epsilon = (1/2, \ldots, 1/2)$ has this property. The scalar product on $\mathbb{C}^n$ defines a constant metric, which is compatible with the natural almost complex structure. Consequently, it is Hermitian and Kähler, and Λ-invariant, so that it descends to a Kähler metric on M. The cohomology groups $H^{p,q}(M)$ are spanned by elements of the form $\mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J$, where I and J are multi-indices of lengths p and q respectively. Thus when n is 2 then the Hodge diamond looks as follows:

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} = \begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 4 & & 1. \\ & 2 & & 2 & \\ & & 1 & & \end{array} \tag{3.1}$$

Viewed as differential manifolds, tori are all diffeomorphic to $(S^1)^{2n}$ which is compact. From the complex point of view, however, if Λ_1 and Λ_2 are two lattices, then the tori they induce are diffeomorphic only if there exists a $\mathbb{C}$-linear bijection f from $\mathbb{C}^n$ to $\mathbb{C}^n$ such that $f(\Lambda_1) = \Lambda_2$.

3.1.2 Projective space

Definition 3.1 *Complex projective space*, denoted $\mathbb{CP}^n$ or just $\mathbb{P}^n$ is defined to be the set of lines in $\mathbb{C}^{n+1}$:

$$\mathbb{P}^n := (\mathbb{C}^{n+1} \backslash \{0\})/\mathbb{C}^*, \tag{3.2}$$

where $\mathbb{C}^*$ acts by multiplication. Its topology is the quotient topology.

Points in $\mathbb{P}^n$ are denoted by *homogeneous coordinates* $(z^0 : \ldots : z^n)$. We construct an atlas consisting of the *standard open covering* and the *inhomogeneous coordinates*:

$$U_\mu := \{(z^0 : \ldots : z^n) \mid z^\mu \neq 0\}, \quad \xi^\nu_{(\mu)} := z^\nu / z^\mu. \tag{3.3}$$

These charts are open and cover the entire space. $\mathbb{P}^n$ is complex by Proposition A.3. It is in fact Kähler, by the Kähler potential:

$$K_\mu := \log\left(\sum_{\nu=0}^{n} |\xi^\nu_{(\mu)}|^2\right). \tag{3.4}$$

On an overlap $U_\mu \cap U_\nu$, we have

$$\log\left(\sum_{\nu=0}^{n} |\xi^\nu_{(\mu)}|^2\right) = \log\left(\left|\frac{z^\kappa}{z^\mu}\right|^2 \sum_{\nu=0}^{n} \left|\frac{z^\nu}{z^\kappa}\right|^2\right) = \log\left(\frac{z^\kappa}{z^\mu}\right) + \log\left(\frac{\bar{z}^\kappa}{\bar{z}^\mu}\right) + K_\kappa.$$

Thus the form ω defined by

$$\omega = i\partial\bar{\partial} K_\mu \tag{3.5}$$

is globally defined. It is straightforward to show that the metric associated to ω is indeed positive definite. This metric is called the *Fubini-Study metric*.

The Hodge diamond of $\mathbb{P}^n$ looks as follows: for even $i \in \{0, \ldots, 2n\}$, $h^{i,i} = 1$ while all other numbers are 0.

There is an obvious line bundle on $\mathbb{P}^n$: the one whose fiber over a point $l \in \mathbb{P}^n$ is the line it represents in $\mathbb{C}^{n+1}$. Explicitly, it is the subset

$$\{(l, x) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \mid x \in l\}.$$

This bundle is called the *tautological line bundle*. Its dual is called the *hyperplane line bundle* and is usually denoted $\mathcal{O}(1)$. It can be shown that every holomorphic line bundle over $\mathbb{P}^n$ is isomorphic to $\mathcal{O}(1)^{\otimes k}$ for some $k \in \mathbb{Z}$. We denote $\mathcal{O}(k) := \mathcal{O}(1)^{\otimes k}$. Thus, the tautological line bundle is $\mathcal{O}(-1)$.

We have the following [12, p. 91]

Proposition 3.2 *For $k \geq 0$, the space of sections from $\mathbb{P}^n$ to $\mathcal{O}(k)$ is canonically isomorphic to the space of homogeneous polynomials of degree k.*

In particular, we see that the homogeneous coordinates are sections of $\mathcal{O}(1)$. Thus the holomorphic tangent bundle $T_{\mathbb{P}^n}$ is spanned by $s_i(z)\frac{\partial}{\partial z_i}$, with the s_i sections of $\mathcal{O}(1)$, but we have to take equivalence classes with respect to overall rescaling, since overall rescaling is trivial in $\mathbb{P}^n$. Thus, there is the exact sequence of holomorphic vector bundles [12, p. 93]

$$0 \to \mathbb{C} \to \mathcal{O}(1)^{\oplus(n+1)} \to T_{\mathbb{P}^n} \to 0. \tag{3.6}$$

This sequence is called the *Euler sequence*. From it and equation (1.5) it follows that

$$K_{\mathbb{P}^n} = \mathcal{O}(-n-1). \tag{3.7}$$

Now consider the total Chern class of $\mathbb{P}^n$. The Euler sequence gives us

$$c(\mathbb{P}^n) = c(T_{\mathbb{P}^n}) = c(\mathcal{O}(1)^{\oplus(n+1)}) = c(\mathcal{O}(1))^{n+1}.$$

Since $\mathcal{O}(1)$ is a line bundle, the expansion of the total Chern class is $c(\mathcal{O}(1)) = 1 + c_1(\mathcal{O}(1))$. Thus if we set $x := c_1(\mathcal{O}(1))$, then

$$c(\mathbb{P}^n) = (1+x)^{n+1}. \tag{3.8}$$

3.1.3 Projective hypersurfaces

Let f be a homogeneous polynomial in $n+1$ complex variables, and assume that 0 is a regular value. If $f(z_0, \ldots, z_n) = 0$, then f is 0 for any representative of $(z_0 : \ldots : z_n) \in \mathbb{P}^n$ since f is homogeneous. Therefore, the subset

$$M := f^{-1}(0)/\mathbb{C}^* \subset \mathbb{P}^n \tag{3.9}$$

is well-defined. By proposition A.3 it is complex, and by Proposition A.10 it is Kähler.

3.2 K3 surfaces

Definition 3.3 A *K3 surface* is defined as a compact complex manifold S of complex dimension 2, having a trivial canonical bundle and such that $h^{1,0}(S) = 0$.

Theorem 3.4 *All K3 surfaces are Kähler, and any two K3 surfaces are diffeomorphic.*

The first part of this theorem is due to Siu [15], and combined with the fact that the canonical bundle is trivial, it implies that K3 surfaces are Calabi-Yau (see Proposition A.13). The second part can be found in [14], and it implies that, for our purposes, we need to explore only a few K3 surfaces.

The fact that the canonical bundle of a K3 surface is trivial implies that there is a nowhere zero holomorphic section. Consider two such sections, s_1 and s_2. Projecting both of them onto the second factor and then taking the quotient gives a globally defined holomorphic function. Then it follows from basic complex analysis that this quotient is constant. Thus the K3 surface admits a single, globally defined, nowhere vanishing holomorphic 2-form, unique up to a constant. It follows that $h^{2,0}(S) = 1$.

3.2.1 Quartics in $\mathbb{P}^3$

Let $(z_0 : \cdots : z_3)$ be projective coordinates on $\mathbb{P}^3$, and consider the hypersurface S defined as the zero-locus of a polynomial p of degree d on $\mathbb{P}^3$. From the Lefschetz hyperplane theorem [13, p. 156] it follows that $h^{1,0} = 0$ of such a hypersurface, and that it is simply connected. Next we will find a d so that $K_S = 0$. For this, we need the following result [12, p. 95].

Proposition 3.5 *Suppose L is a holomorphic line bundle on M. Let X be a smooth hypersurface of a complex manifold M defined by a section $s : M \to L$. Then $N_X \cong L|_X$.*

Now a polynomial p on $\mathbb{P}^n$ of degree d may be regarded as a section from $\mathbb{P}^n$ to $\mathcal{O}(d)$ by Proposition 3.2. Therefore, $N_S \cong \mathcal{O}(d)$, and we obtain the corollary

Corollary 3.6 *If $X \subset \mathbb{P}^n$ is a smooth hypersurface defined by a polynomial on $\mathbb{P}^n$ by a polynomial of degree d, then $K_X \cong \mathcal{O}(d - n - 1)$.*

Thus, S is a K3 surface if and only if $d = 4$.

The normal bundle sequence now becomes

$$0 \to T_S \to T_{\mathbb{P}^n}|_S \to \mathcal{O}(d)|_X, \tag{3.10}$$

implying $c(\mathbb{P}^n) = c(S) \wedge c(\mathcal{O}(d))$. Now $c_1(\mathcal{O}(d)) = dc_1(\mathcal{O}(1)) = dx$ by Proposition 2.76. Thus we obtain

$$(1+x)^4 = c(S) \wedge (1+4x).$$

This implies $c(S) = 1 + 6x^2$, from which it follows that[1] $c_1(S) = 0$ and $c_2(S) = 6x^2$. In particular, S is spin by Proposition 2.96.

Lastly, we calculate the Euler characteristic and signature of S. The holomorphic Euler characteristic is $h^{(0,2)} - h^{(0,1)} + 1 = 2$, so that Noether's formula reads

$$2 = \frac{c_1^2 + c_2}{12} = \frac{\chi}{12},$$

i.e. $\chi = 24$. Its signature is then $\tau = p_1/3 = (c_1^2 - 2c_2)/3 = -\frac{2}{3}24 = -16$.

We now have enough information to compute all the Hodge numbers, $h^{p,q}$. Since $\pi_1(S) = 0$, we have that the first Betti number $b_1(S) = \dim H^1(S) = h^{1,0} + h^{0,1}$ must be zero. The Euler characteristic then fixes $b_2(S)$ which then determines $h^{1,1}$ since we already know $h^{2,0} = 1$ from above. The result is

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} = \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1. \\ & 0 & & 0 & \\ & & 1 & & \end{array} \tag{3.11}$$

Since $b_2 = b_2^+ + b_2^- = 22$ and $\tau = b_2^+ - b_2^- = -16$, we also find $b_2^+ = 3$ and $b_2^- = 19$.

3.2.2 Kummer surfaces

Let $T = \mathbb{C}^2/\Lambda$ be a two-dimensional complex torus. As noted before, this is a Kähler manifold. Now consider the involution σ on it defined by $\sigma(x) = -x$. It is not hard to see that this map has 16 fixed points; for example, when $\Lambda = \mathbb{Z}^4$ then these are the points in which all coordinates are either 0 or $\frac{1}{2}$. Therefore, the quotient space $T/\{\mathrm{id}, \sigma\}$ is not a smooth manifold. However, by replacing each of the points which σ leaves invariant by a copy of $\mathbb{P}^1$ we obtain a new space $\hat{T}$; the quotient of σ on this space does produce a smooth manifold, which will turn out to be K3. These surfaces are called *Kummer surfaces*. We will now make this more precise.

Definition 3.7 Take coordinates $z = (z_1, \ldots, z_n)$ on $\mathbb{C}^n$, and let $(l_1, \ldots, l_n)$ be homogeneous coordinates on $\mathbb{P}^{n-1}$. Now consider the submanifold

$$\widetilde{\mathbb{C}}^n := \{(z,l) \in \mathbb{C}^n \times \mathbb{P}^{n-1} \mid z \in l\}. \tag{3.12}$$

There is an obvious projection $\pi : \widetilde{\mathbb{C}}^n \to \mathbb{C}^n$ which projects onto the first factor. $\widetilde{\mathbb{C}}^n$ along with π is called the *blow-up* of $\mathbb{C}^n$ at 0.

[1] This can also be seen directly by

$$0 = c_1(K_S) = c_1(\det(\Omega^{1,0}(S))) = c_1(\Omega^{1,0}(S)) = -c_1(T_S) = -c_1(S).$$

The relation $z \in l$ may also be written $z_i l_j = z_j l_i$ for all i, j. Note that $\widetilde{\mathbb{C}}^n$ is actually just the line bundle $\mathcal{O}(-1)$.

For each $z \neq 0$ in $\mathbb{C}^n$ there is a single line in $\mathbb{P}^{n-1}$ containing it; that is, $\pi^{-1}(z) = \{l\}$ where l is the line containing z. However, since all lines contain 0, we have $\pi^{-1}(0) = \mathbb{P}^{n-1}$. Thus, the blow-up replaces 0 by a copy of $\mathbb{P}^{n-1}$ (which is why it is called the blow-up at 0). $\pi^{-1}(0) = \mathbb{P}^{n-1}$ is called the *exceptional divisor* and usually denoted by E.

Now if M is an n-dimensional manifold then we can take a chart which maps some $x \in M$ to 0, and apply this construction locally. One then obtains a new complex manifold $\widetilde{M}$ and a holomorphic projection map $\widetilde{M} \to M$. As before, $\widetilde{M}$ and π are called the blow-up of M at x.

The observation that the blow-up replaces the point x by a copy of $\mathbb{P}^n$ is made more formal by the following. If M is an orientable smooth manifold, we denote by $\overline{M}$ the same manifold with the opposite orientation, and if M' is a second manifold, then $M \# M'$ denotes a connected sum.

Proposition 3.8 *Let $x \in M$ be a point in an $(n-1)$-dimensional complex manifold. Then the blow-up $\widetilde{M}$ at x is diffeomorphic as an oriented differentiable manifold to $M \# \overline{\mathbb{P}}^{n-1}$.*

From this it also follows that for $i > 0$,

$$H_i(\widetilde{M}) = H_i(M) \oplus H_i(\mathbb{P}^{n-1}), \tag{3.13}$$

while $H_0(\widetilde{M}) \cong \mathbb{Z}$ if M is connected, since then $\widetilde{M}$ is also connected.

Now we return to the torus T, and the involution σ. Denote the set of points that it leaves fixed by $\mathrm{Fix}(\sigma) \subset T$. Then we can blow up T at each point in $\mathrm{Fix}(\sigma)$ to obtain a new complex manifold $\widetilde{T}$. σ has then a natural extension to $\widetilde{T}$, which is the identity on the $\mathbb{P}^1$-component of $\widetilde{T}$. Denote this map by $\widetilde{\sigma}$. Now we define $S := \widetilde{T}/\tilde{\sigma}$.

Theorem 3.9 *The Kummer surface S is a K3 surface.*

This can be proved by showing three things: first, that S is in fact smooth; second, that it has trivial canonical bundle; and lastly and that $h^{1,0}(S) = 0$. For the first two points, see for example [14, p. 224]. As for the last one, if M is a complex n-dimensional manifold and $\widetilde{M}$ a blow-up, then $H_i(\widetilde{M}) = H_i(M) \oplus H_i(\mathbb{P}^{n-1})$ for $i > 0$. Therefore, by Poincaré-duality $H^1(T) = H^1(\widetilde{T})$. (Since $h^{1,1}(\mathbb{P}^1) = 1$, and since we have to blow up 16 points, this increases $h^{1,1}$ by 16, giving the $h^{1,1} = 20$ that we have already seen in the Hodge diamond of K3 surfaces.) Now by a classical result of Grothendieck [16], the first cohomology group of the quotient S is given by those classes in $H^1(\widetilde{T})$ that are invariant under $\widetilde{\sigma}$, i.e. by those classes in $H^1(T)$ being invariant under σ. However, if $\{x_i\}$ are real coordinates of T then $H^1(T)$ is spanned by the forms $\mathrm{d}x_i$, so there are no classes that σ leaves invariant. Therefore $b_1(S) = 0$. Using the fact that any compact complex surface M having even $b_1(M)$ is Kähler (see e.g. [14, p. 144]), this implies $h^{1,0} = 0$.

It turns out that in complex dimension 2, K3 surfaces are the only Calabi-Yau manifolds that are simply connected and compact. This is our primary reason for studying them.

We shall also need the following facts. If X is a matrix and $X \oplus \cdots \oplus X$ has n summands we simply write nX.

Proposition 3.10 *The intersection form of a K3 surface is of the form $2(-E_8) \oplus 3H$, where H is the matrix with 1's on* *its antidiagonal, and E_8 is the (positive definite) root lattice of the corresponding Lie algebra:*

$$E_8 = \begin{pmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{pmatrix}, \qquad H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.14}$$

Any orientation-preserving diffeomorphism $f : M \to M'$ induces a homomorphism $f^* : H^2(M', \mathbb{Z}) \to H^2(M, \mathbb{Z})$, which necessarily leaves the intersection form invariant. Such a map may be seen as an element of the isometry group $O_Q(3, 19; \mathbb{Z})$ of the intersection form Q. Denote by O_Q^+ the subgroup that preserves the orientation of the positive part H^+.

Theorem 3.11 *The diffeomorphism group of a K3 surface is isomorphic to O_Q^+ ([17–19]).*

Chapter 4

Instantons

In this chapter, we start dealing with physics. Here we specialize the structure group to $G = \mathrm{SU}(2)$. We begin by introducing the action of a gauge theory on Euclidean space, and define the notion of instantons; these are solutions of the action that absolutely minimize it. Then, in the second section, we consider topological invariants on principal bundles, and we shall see that the action of instantons are in fact topological invariants. In the next section, we fix a base space and fiber and find some actual instantons, and in the last section we consider briefly what happens when one considers bundles over $\mathrm{SO}(3)$ instead of $\mathrm{SU}(2)$.

More information on these subjects can be found in [4,9,20], and [5, chapter 10].

4.1 The action

Henceforth, we will assume that M is a four-dimensional manifold.

Definition 4.1 The Euclidean *Yang-Mills action* S_{YM} of a connection is given by[1]

$$S_{\mathrm{YM}} = -\int_M \mathrm{Tr}(F \wedge \star F) = \int_M \langle F, F \rangle \,\mathrm{dVol}. \tag{4.1}$$

Note that it is invariant under gauge transformations. This action is minimized when $\mathcal{D}_A \star F = 0$.

When A_i is a connection which has an (anti-)self-dual field strength (see Definition B.7), we say that the connection is (anti-)self-dual. (Anti-)self-dual connections are solutions of the Yang-Mills action because of the Bianchi identity (Proposition 2.24). However, this may also be seen in another way: note that

$$S_{\mathrm{YM}} = \int_M \langle F, F \rangle \,\mathrm{dVol} = \int_M \langle F_+ + F_-, F_+ + F_- \rangle \,\mathrm{dVol} = \int_M (|F_+|^2 + |F_-|^2) \,\mathrm{dVol}, \tag{4.2}$$

while on the other hand,

$$\begin{aligned} c := \int_M \mathrm{Tr}(F \wedge F) &= -\int_M \langle F, \star F \rangle \,\mathrm{dVol} = -\int_M \langle F_+ + F_-, F_+ - F_- \rangle \,\mathrm{dVol} \\ &= \int_M (|F_-|^2 - |F_+|^2) \,\mathrm{dVol}. \end{aligned} \tag{4.3}$$

(When $G = \mathrm{SU}(n)$ with $n > 1$, this is actually the second Chern class.) Thus the action is bound form below by this number,

$$S_{\mathrm{YM}} \geq |c|,$$

[1] For an introduction of the Hodge star, see Appendix B.

with equality if and only if $F = \star F$ or $F = -\star F$, in which case $S_{\text{YM}} = \mp c$. These solutions are called *instantons*. They are of mathematical interest, because c and with it the action of instantons is an invariant of the bundle. Physically they are interesting because, among other reasons, they are easier to find than general solutions of the action, because $F = \pm\star F$ is a differential equation of order 1, while $\mathcal{D}_A \star F = 0$ is of order 2.

The pseudo-Riemannian case

The Euclidian action (4.1) is usually obtained from the Minkowski-action by a Wick rotation, sending $t = x_0$ to ix^0. Thus, we can retrieve the Minkowski-action by doing the opposite rotation: $x^4 \to -ix^4 =: -ix^0$. Writing the Euclidian action as

$$S^{\text{E}}_{\text{YM}} = \frac{1}{4}\int F^a_{\mu\nu}F^{\mu\nu,a}\,\mathrm{dVol} = \frac{1}{4}\int F^a_{\mu\nu}F^{\mu\nu,a}\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^4 \tag{4.4}$$

the action in Minkowski-spacetime becomes

$$\begin{aligned} S^{\text{M}}_{\text{YM}} &= -\frac{i}{4}\int F^a_{\mu\nu}F^{\mu\nu,a}\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^0 = \frac{i}{4}\int F^a_{\mu\nu}F^{\mu\nu,a}\,\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^3 \\ &= -i\int_M \mathrm{Tr}(F \wedge \star F). \end{aligned} \tag{4.5}$$

However, the Minkowski action is usually defined as just $-\int_M \mathrm{Tr}(F \wedge \star F)$, while the factor i is put in the path integral, $\int [DA] e^{iS^{\text{M}}_{\text{YM}}/\hbar}$.

4.2 SU(2)-bundles over $\mathbb{R}^4$

Let us now focus on SU(2)-bundles over $\mathbb{R}^4$. (For various descriptions of SU(2), see section C.2 in the appendix.) Since $\mathbb{R}^4$ is not compact, we will have to be careful with the convergence of the integrals. We will look at connections which have finite Yang-Mills action. In particular, this means that the field strength vanishes sufficiently fast at infinity.

Euclidean space $\mathbb{R}^4$ is conformally equivalent to the 4-sphere S^4 with a point removed. Furthermore, the Hodge dual is conformally invariant, so the (anti-)self-duality condition is conformally invariant as well. Hence if an instanton on $\mathbb{R}^4$ has finite action and it extends to the point at infinity, it defines an instanton on S^4.

We can cover S^4 by two hemispheres, U_N and U_S, say. Then there is only one transition map $g : U_N \cap U_S \to \mathrm{SU}(2)$. However, both $U_N \cap U_S$, and SU(2) are homeomorphic to S^3, so we can classify $P(S^4, \mathrm{SU}(2))$-bundles if we can classify all maps

$$g : S^3 \to S^3.$$

This classification is achieved by $\pi_3(S^3) = \mathbb{Z}$. The element of $\mathbb{Z}$ that corresponds to a given $g : S^3 \to S^3$ is called the *degree* or *winding number* of g.

To be specific, let σ_i be the Pauli matrices, and I the identity matrix in SU(2). Denote an element $x \in S^3$ by $(x_1, \ldots, x_4)$. g_0 is the trivial map sending all x to $e \in \mathrm{SU}(2)$. The diffeomorphism $S^3 \approx \mathrm{SU}(2)$ is achieved by the map

$$g_1 : x \mapsto \frac{1}{|x|}(x^4 I + ix^i\sigma_i). \tag{4.6}$$

Being the identity map, this map obviously has degree 1. The map corresponding to degree n is

$$g_n = (t_1)^n : x \mapsto \frac{1}{|x|^n}(x^4 I + ix^i\sigma_i)^n. \tag{4.7}$$

Now, the Lie algebra of SU(n) consists of traceless skew-hermitian matrices. Hence $c_1(F) = \frac{i}{2\pi}\operatorname{Tr}(F) = 0$, and

$$c_2(F) = \frac{1}{8\pi^2}\operatorname{Tr}(F \wedge F); \tag{4.8}$$

i.e., up to an unimportant factor, the Yang-Mills action of self-dual (anti-self-dual) connections is equal to second Chern-number:

$$S_{\mathrm{YM}} = \mp 8\pi^2 c_2 = \mp \int_{S^4} \operatorname{Tr}(F \wedge F). \tag{4.9}$$

To calculate this integral, note that

$$\int_{S^4} c_2(F) = \int_{U_N} \mathrm{d}Q_3(A,F) + \int_{U_S} \mathrm{d}Q_3(A',F') = \int_{S^3} Q_3(A,F) - Q_3(A',F'),$$

where Q_3 is the Chern-Simons form associated to $c_2(F)$. So our first task is to find this Chern-Simons form. Writing $A^n = \underbrace{A \wedge \cdots \wedge A}_{n}$, we have

$$\begin{aligned} Q_3(A,F) &= \frac{1}{4\pi^2}\int_0^1 \mathrm{d}t\ \operatorname{Tr}(A \wedge F_t) = \frac{1}{4\pi^2}\int_0^1 \mathrm{d}t\ \operatorname{Tr}(tA \wedge \mathrm{d}A + t^2 A^3) \\ &= \frac{1}{8\pi^2}\operatorname{Tr}(A \wedge \mathrm{d}A + \frac{2}{3}A^3) = \frac{1}{8\pi^2}\operatorname{Tr}(A \wedge F - \frac{1}{3}A^3). \end{aligned} \tag{4.10}$$

Denoting with $X \sim Y$ the equivalence relation $\operatorname{Tr}(X) = \operatorname{Tr}(Y)$, we have

$$\begin{aligned} A' \wedge F' &= (g^{-1}Ag + g^{-1}\mathrm{d}g) \wedge (g^{-1}Fg) = g^{-1}A \wedge Fg + g^{-1}\mathrm{d}g \wedge g^{-1}Fg \\ &\sim A \wedge F + g^{-1}F \wedge \mathrm{d}g = A \wedge F + g^{-1}\mathrm{d}A \wedge \mathrm{d}g + g^{-1}A \wedge A \wedge \mathrm{d}g \end{aligned}$$

and

$$A'^3 \sim A^3 + 3g^{-1}A \wedge A \wedge \mathrm{d}g + 3g^{-1}A \wedge \mathrm{d}g \wedge g^{-1}\mathrm{d}g + (g^{-1}\mathrm{d}g)^3$$

so

$$\begin{aligned} c_2 &= \int_{S^3} Q_3(A,F) - Q_3(A',F') \\ &= \frac{1}{8\pi^2}\int_{S^3} \operatorname{Tr}\Big(- g^{-1}\mathrm{d}A \wedge \mathrm{d}g - g^{-1}A \wedge A \wedge \mathrm{d}g + g^{-1}A \wedge A \wedge \mathrm{d}g \\ &\qquad + g^{-1}A \wedge \mathrm{d}g \wedge g^{-1}\mathrm{d}g + \frac{1}{3}(g^{-1}\mathrm{d}g)^3\Big) \\ &= \frac{1}{8\pi^2}\int_{S^3} \operatorname{Tr}\left(-g^{-1}\mathrm{d}A \wedge \mathrm{d}g + g^{-1}A \wedge \mathrm{d}g \wedge g^{-1}\mathrm{d}g + \frac{1}{3}(g^{-1}\mathrm{d}g)^3\right) \\ &= \frac{1}{8\pi^2}\int_{S^3} \operatorname{Tr}\left(-\mathrm{d}(g^{-1}A \wedge \mathrm{d}g) + \frac{1}{3}(g^{-1}\mathrm{d}g)^3\right) \\ &= \frac{1}{24\pi^2}\int_{S^3} \operatorname{Tr}(g^{-1}\mathrm{d}g)^3, \end{aligned} \tag{4.11}$$

in which we used the identities $\operatorname{Tr}(\alpha \wedge \beta \wedge \gamma) = \operatorname{Tr}(\gamma \wedge \alpha \wedge \beta)$, $\operatorname{Tr}(\cdots \wedge \alpha g \wedge \beta \wedge \cdots) = \operatorname{Tr}(\cdots \wedge \alpha \wedge g\beta \wedge \cdots)$, and $\mathrm{d}(g^{-1}) = -g^{-1}\mathrm{d}g\, g^{-1}$. This is exactly equal to the degree of the transition map g. Summarizing, in the case of self-dual (anti-self-dual) connections, we have

$$S_{\mathrm{YM}} = \mp 8\pi^2 \deg(g) =: \pm 8\pi^2 n, \tag{4.12}$$

where n is called the *instanton number*. In particular, since we know that the action is positive definite, it follows that on bundles for which $\deg(g) > 0$ (or equivalently $n < 0$), only anti-self-dual connections exist, and on bundles which have $\deg(g) < 0$ and $n > 0$ only self-dual connections exist. If $\deg(g) = 0$, i.e. the bundle is trivial, then the instanton-action is 0. But since the Yang Mills-action is just the inner product of F with itself, this implies $F = 0$, i.e. there are no instantons in this case. Also note that if G is abelian, then $F = \mathrm{d}A + \frac{1}{2}[A, A] = \mathrm{d}A$. In this case, (anti-)self-dual field strengths are co-exact: $F = \star F = \star \mathrm{d}A = \star \mathrm{d} \star^2 A = \mathrm{d}^\dagger \star A$. Then $(F, F) = (\mathrm{d}A, \mathrm{d}^\dagger \star A) = (\mathrm{d}^2 A, \star A) = 0$, so $F = 0$; so there are no instantons in this case.[2]

4.2.1 Actual instantons

For SU(n) there is a certain construction, called the ADHM construction ([21], see also [9,22]), which gives a correspondence between all instantons and certain systems of finite-dimensional algebraic data. However, this is a rather technical approach, so we shall not discuss it here. Instead, we shall focus on SU(2) and find some instantons (not all of them) through a certain ansatz.

We first introduce some symbols. (Note that since we are working in a Euclidian space, there is no distinction between upper and lower indices, so we will be sloppy with the location of the indices.) (For a more comprehensive introduction in instantons, see [20]). We will denote the Pauli matrices by σ_a, $a = 1, \ldots, 3$. We also define $\sigma_4 = iI$ and write $\sigma_\mu = (\sigma_a, iI)$. These then satisfy

$$\frac{1}{2}(\sigma_\mu \sigma_\nu^\dagger - \sigma_\nu \sigma_\mu^\dagger) = \delta_{\mu\nu} I, \tag{4.13}$$

as may be derived by using $\{\sigma_a, \sigma_b\} = 2\delta_{ab} I$ and the unitarity of σ_4.

Definition 4.2 The *'t Hooft symbol*, introduced in introduced in [23] and denoted ${\eta^a}_{\mu\nu}$, is defined by:

$${\eta^a}_{\mu\nu} = {\epsilon^a}_{\mu\nu} + \delta^a_\mu \delta^4_\nu - \delta^a_\nu \delta^4_\mu. \tag{4.14}$$

In addition, we define an antisymmetric 't Hooft symbol ${\bar\eta^a}_{\mu\nu}$ by

$${\bar\eta^a}_{\mu\nu} = {\epsilon^a}_{\mu\nu} - \delta^a_\mu \delta^4_\nu + \delta^a_\nu \delta^4_\mu. \tag{4.15}$$

and

$$\bar\sigma_{\mu\nu} = i{\eta^a}_{\mu\nu}\sigma_a \quad \text{and} \quad \sigma_{\mu\nu} = i{\bar\eta^a}_{\mu\nu}\sigma_a. \tag{4.16}$$

Proposition 4.3 *The 't Hooft symbol satisfies the following properties:*

$$\begin{aligned} {\eta^a}_{\mu\nu} &= \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}{\eta^a}_{\rho\sigma} \quad \textit{(i.e., it is self-dual)}, \\ {\epsilon^a}_{bc}{\eta^b}_{\mu\kappa}{\eta^c}_{\nu\lambda} &= \delta_{\mu\nu}{\eta^a}_{\kappa\lambda} + \delta_{\kappa\lambda}{\eta^a}_{\mu\nu} - \delta_{\mu\lambda}{\eta^a}_{\kappa\nu} - \delta_{\kappa\nu}{\eta^a}_{\mu\lambda}, \\ \eta_{a\mu\nu}\eta_{a\mu\nu} &= 12. \end{aligned} \tag{4.17}$$

$\bar\eta$ is anti-self-dual, and satisfies the lower two identities. Furthermore,

$$\begin{aligned} \sigma_{\mu\nu} &= \frac{1}{2}(\sigma_\mu \sigma_\nu^\dagger - \sigma_\nu \sigma_\mu^\dagger), \quad \bar\sigma_{\mu\nu} = \frac{1}{2}(\sigma_\mu^\dagger \sigma_\nu - \sigma_\nu^\dagger \sigma_\mu), \\ [\sigma_{\mu\kappa}, \sigma_{\nu\lambda}] &= -2(\delta_{\mu\nu}\sigma_{\kappa\lambda} + \delta_{\kappa\lambda}\sigma_{\mu\nu} - \delta_{\mu\lambda}\sigma_{\kappa\nu} - \delta_{\kappa\nu}\sigma_{\mu\lambda}), \\ \epsilon_{\mu\nu\kappa\lambda}\sigma_{\lambda\tau} &= \delta_{\mu\tau}\sigma_{\nu\kappa} - \delta_{\nu\tau}\sigma_{\mu\kappa} + \delta_{\kappa\tau}\sigma_{\mu\nu}. \end{aligned} \tag{4.18}$$

[2] Since this argument uses the inner product, which is defined in terms of an integral over M, this integral needs to exist for this F. However, since the point of instantons is to absolutely minimize the action, this does not seem a very serious restriction. When we have this restriction, though, there is another argument that goes much further: If F is a minimum of the action (not necessarily an instanton), then it is simultaneously exact and harmonic. If the bundle is trivial and if M is Riemannian and without boundary (such as $\mathbb{R}^4$), then by the Hodge decomposition, F is zero. Thus, no nonzero Yang Mills fields for which the action is finite exist *at all* these cases.

To find the instantons, we will use a certain ansatz:

$$A_\mu = \frac{1}{2}\sigma_{\mu\nu}\partial_\nu \ln(\phi(x^2)). \tag{4.19}$$

The field strength of this is

$$\begin{aligned}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \\
&= \frac{1}{2}\sigma_{\nu\kappa}\partial_\mu\partial_\kappa \ln\phi - (\mu \leftrightarrow \nu) + \frac{1}{4}[\sigma_{\mu\kappa}, \sigma_{\nu\lambda}]\partial_\kappa \ln\phi\, \partial_\lambda \ln\phi \\
&= \frac{1}{2}\sigma_{\nu\kappa}\partial_\mu\partial_\kappa \ln\phi - (\mu \leftrightarrow \nu) + \frac{1}{2}\sigma_{\mu\lambda}\partial_\nu \ln\phi\, \partial_\lambda \ln\phi - (\mu \leftrightarrow \nu) - \frac{1}{2}\sigma_{\mu\nu}(\partial \ln\phi)^2 \\
&= \frac{1}{2}\sigma_{\nu\kappa}(\partial_\kappa\partial_\mu \ln\phi - \partial_\kappa \ln\phi\, \partial_\mu \ln\phi) - (\mu \leftrightarrow \nu) - \frac{1}{2}\sigma_{\mu\nu}(\partial \ln\phi)^2
\end{aligned}$$

The Hodge dual of this is

$$\begin{aligned}
\star F_{\mu\nu} &= \frac{1}{2}\epsilon_{\mu\nu\kappa\lambda}F_{\kappa\lambda} \\
&= \frac{1}{2}\sigma_{\nu\kappa}(\partial_\kappa\partial_\mu \ln\phi - \partial_\kappa \ln\phi\, \partial_\mu \ln\phi) - (\mu \leftrightarrow \nu) + \frac{1}{2}\sigma_{\mu\nu}(\partial^2 \ln\phi).
\end{aligned}$$

Focusing on self-dual instantons for the moment, equating these two and comparing coefficients of the σ matrices, we find one equation that is satisfied trivially, and one other equation:

$$-(\partial \ln\phi)^2 = \partial^2 \ln\phi. \tag{4.20}$$

Now,

$$-(\partial \ln\phi)^2 = -\partial_\mu \ln\phi\, \partial_\mu \ln\phi = -\frac{1}{\phi^2}\partial_\mu\phi\, \partial_\mu\phi$$

and

$$\partial^2 \ln\phi = \partial_\mu\partial_\mu \ln\phi = \partial_\mu \left(\frac{1}{\phi}\partial_\mu\phi\right) = -\frac{1}{\phi^2}\partial_\mu\phi\, \partial_\mu\phi + \frac{1}{\phi}\partial_\mu\partial_\mu\phi$$

so we conclude that under our ansatz, the self-duality equation $F = \star F$ is satisfied if

$$\frac{\partial^2\phi}{\phi} = 0. \tag{4.21}$$

This equation is satisfied by $\frac{1}{x^2}$:

$$\partial_\mu\partial_\mu \frac{1}{x^2} = \partial_\mu \left(\frac{-2x^\mu}{x^4}\right) = \frac{-2\partial_\mu x^\mu x^4 + 8x^2 x^\mu x^\mu}{x^8} = 0,$$

since $\partial_\mu x^\mu = 4$. A more general solution is

$$\phi(x) = 1 + \sum_{i=1}^{n} \frac{\rho_i^2}{(x - x_i)^2}. \tag{4.22}$$

The 1 is to ensure that when we plug this in $A_\mu = \frac{1}{2}\sigma_{\mu\nu}\partial_\nu \ln\phi$, A_μ goes to zero as $|x|$ goes to infinity. Let us see what this solution looks like for $n = 1$:

$$A_\mu^{\text{sing}} = \frac{1}{2}\sigma_{\mu\nu}\partial_\nu \ln\left(1 + \frac{\rho^2}{(x - x_1)^2}\right)$$

$$
\begin{aligned}
&= \frac{1}{2}\sigma_{\mu\nu}\left(1+\frac{\rho^2}{(x-x_1)^2}\right)^{-1}\cdot\frac{-2\rho^2(x-x_1)^\nu}{(x-x_1)^4} \\
&= -\frac{\rho^2}{(x-x_1)^2+\rho^2}\,\sigma_{\mu\nu}\frac{(x-x_1)^\nu}{(x-x_1)^2}
\end{aligned} \tag{4.23}
$$

This expression is singular at $x = x_1$. Fortunately, however, all is not lost; the singularity is not intrinsic to this connection, in the sense that it can be gauge-transformed away. Let us define a gauge by

$$
U(x) = \frac{x_4 I + i x_k \sigma_k}{\sqrt{x^2}} = \frac{i\sigma_\mu^\dagger x^\mu}{\sqrt{x^2}}. \tag{4.24}
$$

U indeed satisfies $U(x)^\dagger = U^{-1}(x)$ so $U(x) \in \mathrm{SU}(2)$. We calculate

$$
\begin{aligned}
U(x)^{-1}\partial_\mu U(x) &= -\frac{i\sigma_\nu x^\nu}{\sqrt{x^2}}\partial_\mu\frac{i\sigma_\lambda^\dagger x^\lambda}{\sqrt{x^2}} = -\frac{i\sigma_\nu x^\nu}{\sqrt{x^2}}\left(\frac{i\sigma_\mu^\dagger}{\sqrt{x^2}} - \frac{i\sigma_\lambda^\dagger x^\lambda x_\mu}{\sqrt{x^2}^{\,3}}\right) \\
&= \frac{\sigma_\nu\sigma_\mu^\dagger x^\nu}{x^2} - \frac{\sigma_\nu\sigma_\lambda^\dagger x^\nu x^\lambda x_\mu}{x^4} \\
&= \frac{\frac{1}{2}(\sigma_\nu\sigma_\mu^\dagger - \sigma_\mu\sigma_\nu^\dagger) + \frac{1}{2}(\sigma_\nu\sigma_\mu^\dagger + \sigma_\mu\sigma_\nu^\dagger)}{x^2}x^\nu \\
&\quad - \frac{\frac{1}{2}(\sigma_\nu\sigma_\lambda^\dagger - \sigma_\lambda\sigma_\nu^\dagger) + \frac{1}{2}(\sigma_\nu\sigma_\lambda^\dagger + \sigma_\lambda\sigma_\nu^\dagger)}{x^4}x^\nu x^\lambda x_\mu \\
&= \frac{\sigma_{\nu\mu} + \delta_{\mu\nu} I}{x^2}x^\nu - \frac{\sigma_{\nu\lambda} + \delta_{\nu\lambda} I}{x^4}x^\nu x^\lambda x_\mu \\
&= -\sigma_{\mu\nu}\frac{x^\nu}{x^2} + \frac{x_\mu I}{x^2} - 0 - \frac{x^2 x_\mu I}{x^4} \\
&= -\sigma_{\mu\nu}\frac{x^\nu}{x^2}.
\end{aligned} \tag{4.25}
$$

The third term is zero because in it the antisymmetric matrix $\sigma_{\nu\lambda}$ is multiplied by the symmetric expression $x^\nu x^\lambda$. Since $U(x)$ is unitary, we also have

$$
U(x)\partial_\mu U^{-1}(x) = -\bar{\sigma}_{\mu\nu}\frac{x^\nu}{x^2}.
$$

Now note that we can write (4.23) as

$$
A_\mu^{\mathrm{sing}} = \frac{\rho^2}{(x-x_1)^2+\rho^2}\,U(x-x_1)^{-1}\partial_\mu U(x-x_1),
$$

so that we can transform the pure gauge on the right away by the inverse gauge transformation:

$$
\begin{aligned}
A_\mu &= U A_\mu^{\mathrm{sing}} U^{-1} + U\partial_\mu U^{-1} = \frac{\rho^2}{(x-x_1)^2+\rho^2}UU^{-1}\partial_\mu UU^{-1} - UU^{-1}\partial_\mu UU^{-1} \\
&= \partial_\mu UU^{-1}\left(\frac{\rho^2}{(x-x_1)^2+\rho^2} - 1\right) = U\partial_\mu U^{-1}\left(\frac{(x-x_1)^2}{(x-x_1)^2+\rho^2}\right) \\
&= -\bar{\sigma}_{\mu\nu}\frac{(x-x_1)^\nu}{(x-x_1)^2+\rho^2}, \quad n = 1.
\end{aligned} \tag{4.26}
$$

in which we used the identity $\mathrm{d}U^{-1} = -U^{-1}\mathrm{d}UU^{-1}$ various times.

Anti-self-dual instantons may be found by taking the ansatz (4.19) with $\bar{\sigma}$ instead of σ. Then the entire calculation goes through in almost exactly the same way, and we find for $n = -1$

$$
A_\mu = -\sigma_{\mu\nu}\frac{(x-x_1)^\nu}{(x-x_1)^2+\rho^2}, \quad n = -1. \tag{4.27}
$$

The only difference between instantons and anti-instantons is that $\bar{\sigma}$ gets changed to σ.

4.2.2 The $n = 1$ instanton

Let us check by an explicit calculation for the one-instanton that it indeed has $n = 1$. This instanton is sometimes also called the BPST instanton, after the authors of the article that introduced it ([24]). We may write it as

$$A^a_\mu = \frac{2\eta^a{}_{\mu\nu}(x-x_1)^\nu}{(x-x_1)^2+\rho^2} \tag{4.28}$$

where x_1 and ρ are arbitrary parameters. They correspond to the position and size of the instantons. In order to calculate the field strength, note that equation (2.15) becomes in coordinates

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + [A_\mu, A_\nu]^a = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^a{}_{bc} A^b_\mu A^c_\nu \tag{4.29}$$

where ϵ is the totally antisymmetric tensor (which are the structure constants of SU(2)). Now,

$$\begin{aligned}
\partial_\mu A^a_\nu &= \frac{2\eta^a{}_{\nu\kappa}\partial_\mu(x-x_1)^\kappa[(x-x_1)^2+\rho^2] - 2\eta^a{}_{\nu\kappa}(x-x_1)^\kappa 2(x-x_1)_\mu}{[(x-x_1)^2+\rho^2]^2} \\
&= \frac{2\eta^a{}_{\nu\mu}[(x-x_1)^2+\rho^2] - 4\eta^a{}_{\nu\kappa}(x-x_1)^\kappa(x-x_1)_\mu}{[(x-x_1)^2+\rho^2]^2},
\end{aligned}$$

and

$$\begin{aligned}
\epsilon^a{}_{bc}A^b_\mu A^c_\nu &= \frac{4\epsilon^a{}_{bc}\eta^b{}_{\mu\kappa}\eta^c{}_{\nu\lambda}(x-x_1)^\kappa(x-x_1)^\lambda}{[(x-x_1)^2+\rho^2]^2} \\
&= \frac{4(\delta_{\mu\nu}\eta^a{}_{\kappa\lambda} + \delta_{\kappa\lambda}\eta^a{}_{\mu\nu} - \delta_{\mu\lambda}\eta^a{}_{\kappa\nu} - \delta_{\kappa\nu}\eta^a{}_{\mu\lambda})(x-x_1)^\kappa(x-x_1)^\lambda}{[(x-x_1)^2+\rho^2]^2} \\
&= \frac{4\eta^a{}_{\mu\nu}(x-x_1)^2 - 4\eta^a{}_{\mu\lambda}(x-x_1)^\nu(x-x_1)^\lambda + 4\eta^a{}_{\nu\kappa}(x-x_1)^\kappa(x-x_1)^\mu}{[(x-x_1)^2+\rho^2]^2}
\end{aligned}$$

(In the second line, the first term in the numerator vanishes trivially.) This gives

$$F^a_{\mu\nu} = \frac{-4\eta^a{}_{\mu\nu}\rho^2}{[(x-x_1)^2+\rho^2]^2}. \tag{4.30}$$

Since F is a multiple of the 't Hooft symbol, it is obviously self-dual. As for the instanton number, let us take $T_i = -\frac{i}{2}\sigma_i$ as generators for the Lie algebra $\mathfrak{su}(2)$, where σ_i are the Pauli matrices. Then $\mathrm{Tr}(T_a T_b) = -\frac{1}{2}\delta_{ab}$, so

$$\begin{aligned}
\mathrm{Tr}(F_{\mu\nu}F_{\mu\nu}) &= F^a_{\mu\nu}F^b_{\mu\nu}\,\mathrm{Tr}(T_aT_b) = -\frac{1}{2}F^a_{\mu\nu}F^a_{\mu\nu} \\
&= -8\eta^a{}_{\mu\nu}\eta^a{}_{\mu\nu}\frac{\rho^4}{[(x-x_1)^2+\rho^2]^4} \\
&= -96\frac{\rho^4}{[(x-x_1)^2+\rho^2]^4},
\end{aligned}$$

and

$$S_{\mathrm{YM}} = -\frac{1}{2}\int \mathrm{d}^4x\,\mathrm{Tr}(F_{\mu\nu}F_{\mu\nu}) = 48\rho^4\int \mathrm{d}^4x\frac{1}{[(x-x_1)^2+\rho^2]^4} = 48\pi^2\frac{1!1!}{3!} = 8\pi^2, \tag{4.31}$$

in which we used the integral

$$\int \mathrm{d}^4x\,\frac{(x^2)^n}{(x^2+\rho^2)^m} = \pi^2(\rho^2)^{n-m+2}\frac{\Gamma(n+2)\Gamma(m-n-2)}{\Gamma(m)}.$$

So we indeed have $n = 1$.

4.3 SO(3)-bundles

When considering Yang-Mills theories over K3 later on, we shall also have to deal with SO(3) bundles. These are slightly more complicated to deal with, because SO(3) is not simply connected. We have remarked in Chapter 2 that an SO(3)-bundle E is fully determined by $p_1(E)$ and $w_2(E)$. However, in this case the instanton number n no longer needs to be an integer. We will sketch a (rather abstract) argument that shows that it always is a multiple of $\frac{1}{4}$.

It is enough to consider the case of universal bundles, i.e. $M = \mathrm{BSU}(2)$, and the defining representations of SU(2) and SO(3) on $\mathbb{C}^2$ and $\mathbb{R}^3$ respectively. The maximal torus of SU(2) consists matrices of the form $\mathrm{diag}(e^{i\theta}, e^{-i\theta})$, so there is a group homomorphism $\mathrm{U}(1) \to \mathrm{SU}(2)$, $z \mapsto \mathrm{diag}(z, z^{-1})$. (Note that for $z \in \mathrm{U}(1)$ we have $z^{-1} = \bar{z}$.) Going over to the classifying spaces, this inclusion induces a map $\mathrm{BU}(1) \to \mathrm{BSU}(2)$ which is injective in cohomology. Applying the splitting principle along this particular map to an SU(2)-bundle V, it follows that V decomposes into a bundle of the form $L \oplus L^{-1}$ for some L. Denote $u := c_1(L)$, then

$$c(V) = (1+u)(1-u) = 1 - u^2,$$

i.e. $c_2(V) = -c_1(L)^2$.

Now take a SO(3)-bundle E and suppose it is liftable to an SU(2)-bundle V. Then it follows from the representation theory of SU(2) that E decomposes as $L^2 \oplus \mathbb{R}$, which has complexification $(L^2 \oplus \mathbb{R}) \otimes \mathbb{C} = (L^2 \otimes \mathbb{C}) \oplus (\mathbb{R} \otimes \mathbb{C}) = L^2 \oplus L^{-2} \oplus \mathbb{C}$. Then since

$$c(E \otimes \mathbb{C}) = c(L^2 \oplus L^{-2} \oplus \mathbb{C}) = (1+2u)(1-2u) = 1 - 4u^2,$$

we find

$$p_1(E) = -c_2(E \otimes \mathbb{C}) = 4u^2 = -4c_2(V). \tag{4.32}$$

What about when E is not liftable to an SU(2)-bundle in this way? Recall that we first encountered the integral $\int_M \mathrm{Tr}(F^2)$, and only then identified this with either c_2 or p_1. However, even if E is not liftable, then in the context of gauge theories one still deals with the above integral. Since the form F takes its values in the relevant Lie algebras which are isomorphic, any SO(3)-connection induces an SU(2)-connection. For such a connection the above argument applies. For SO(3)-bundles, then, the instanton number therefore becomes

$$n = \frac{1}{8\pi^2} \int_M \mathrm{Tr}(F^2) = -\frac{1}{4} p_1 \tag{4.33}$$

which is a multiple of $\frac{1}{4}$ since p_1 is an integer. When M is spin, it is in fact a multiple of $\frac{1}{2}$ (see section 2.5.6).

Chapter 5

The instanton moduli space

In this chapter we define and deal with moduli spaces. We shall first restrict our attention to a class of connections which are 'nice' in a certain way, and then consider two of these to be equivalent when they are related to each other by a gauge transformation. The result of this is a moduli space. These have various interesting applications in both mathematics and physics; to us they will mainly be important because they play a role in the path integral of quantummechanical gauge theories. General references for this chapter are [9,22,25,26]. Much of its contents were first introduced in [27].

Denote the space of irreducible connections by $\hat{\mathscr{A}}$ and let $\tilde{\mathscr{G}} := \mathscr{G}/Z(\mathscr{G})$.

Definition 5.1 Let E be any Riemannian vector bundle with connection over a compact Riemannian manifold M. For each positive integer $k \geq 0$ we can define a *Sobolev k-norm* $\|\cdot\|_k$ on $\Omega^0(M, E)$ by

$$\|\phi\|_k := \int_M \left(\sum_{i=0}^{k} \|\nabla^i \phi\|^2 \right) \mathrm{dVol}. \tag{5.1}$$

The completion of $\Omega^0(M, E)$ with respect to this norm is a Hilbert space that we will denote by $\Omega^0_k(M, E)$. If M is compact, then it may be shown that the inner product, and hence the norm, does not depend on the metrics on M and E and the connection.

The space $\Omega^p(M, E)$ is just the space of sections of the bundle $\bigwedge^p T^*M \otimes E$, which has a metric induced by those of M and E. Thus we can define a Sobolev norm on this space as well, and the completion with respect to this norm is denoted $\Omega^p_k(M, E)$.

In general, for Sobolev spaces $H_k(M)$ we have the embeddings $H_k(M) \subset H_l(M)$ for $k > l$.

Other spaces of maps can be given a Sobolev-completion by this as well. For example, the space of maps from a manifold M to another N, may be seen as sections to the trivial bundle $M \times N \to M$. In particular, the gauge group, considered as a subspace of endomorphisms of P, can be given a Sobolev k-norm and completion, which we will denote by $\mathscr{G}_k$. Furthermore, if we choose a base connection $\omega_0 \in \mathscr{A}$, then $\mathscr{A}_k := \omega_0 + \Omega^1_k(M, \mathrm{ad}\, P)$.

For $k \geq 2$, the action $\mathscr{G}_{k+1} \times \mathscr{A}_k \to \mathscr{A}_k$ is smooth.

5.1 The moduli space

Definition 5.2 Let $\hat{\mathscr{A}}^+$ be the space of self-dual irreducible connections. Then the *moduli space* is defined to be

$$\mathscr{M}_k := \hat{\mathscr{A}}^+_k / \tilde{\mathscr{G}}_{k+1}. \tag{5.2}$$

endowed with the quotient topology. We denote the orbit space of $\hat{\mathscr{A}}$ under $\tilde{\mathscr{G}}$ by

$$\mathscr{B}_k := \hat{\mathscr{A}}_k / \tilde{\mathscr{G}}_{k+1}. \tag{5.3}$$

From a physical point of view, this definition also makes sense, since two connections which are equivalent up to a gauge transformation represent the same physical entity.

Theorem 5.3 *For $k \geq 2$, the natural inclusion of $\mathscr{M}_{k+1}$ in $\mathscr{M}_k$ is a homeomorphism.*

(See [9, Proposition 4.2.16]). Thus, henceforth we will drop the Sobolev subscripts.

Denote the formal adjoint of $\mathcal{D}_A$ by $\mathcal{D}_A^\dagger$, another important theorem is ([9, p. 132], [26, p. 33])

Theorem 5.4 *The space $\mathscr{B}_k$ is a smooth Hilbert manifold with local charts given by $\pi : \mathcal{O}_{A,\epsilon} \to \mathscr{B}_k$ where*

$$\mathcal{O}_{A,\epsilon} := \left\{ a \in \Omega^1(M, \operatorname{ad} P) \mid \mathcal{D}_A^\dagger a = 0, \|a\|_{L^2_{k-1}} < \epsilon \right\} \tag{5.4}$$

and $\epsilon > 0$.

Suppose M is any manifold, and G a group acting on it. Then in general we have that M/G is Hausdorff if and only if the graph of the action $\{(x, g \cdot x) \mid x \in M, g \in G\} \subset M \times M$ is closed. Furthermore, if the action is free, and if through each $x \in M$ we can find a *slice* of the action – that is, an open submanifold N containing x such that

(a) $T_y M = T_y(G \cdot y) \oplus T_y N$ for all $y \in n$,

(b) the restriction of the projection $M \to M/G$ to N is injective,

then M/G is a manifold.

Motivated by this, we will consider the tangent space of $\mathscr{G} \cdot A$ (where $A \in \mathscr{A}$, suppressing the local indices). $\mathscr{A}$ being an affine space over $\Omega^1(M, \operatorname{ad} P)$, the tangent space $T_A\mathscr{A}$ can naturally be identified with $\Omega^1(M, \operatorname{ad} P)$. Furthermore, the Lie algebra of $\mathscr{G}$ is $\mathfrak{G} := T_1\mathscr{G} = T_1 C^\infty(M, \operatorname{Ad} P) = C^\infty(M, \operatorname{ad} P)$.

Proposition 5.5 *Under these identifications, the differential at 1 of the action of $\mathscr{G}$ on a connection A is the map $\mathcal{D}_A$. In particular, the subspace $\operatorname{im}(\mathcal{D}_A) \subset \Omega^1(M, \operatorname{ad} P)$ represents the tangent space to the orbit $\mathscr{G} \cdot A$.*

Proof Let $\Theta \in \mathfrak{G}$, and represent it with a family of local functions $\theta_i : U_\alpha \to \mathfrak{g}$. Then if A_i is a connection and $\exp : \mathfrak{G} \to \mathscr{G}$ the pointwise exponential map, the action of the curve $\exp(t\theta_i)$ on A_i is given by

$$A_i^{\exp(t\theta_i)} = \exp(t\theta_i) A_i \exp(-t\theta_i) - \exp(-t\theta_i)\, \mathrm{d} \exp(t\theta_i).$$

Differentiating with respect to t one obtains after some work

$$\frac{\mathrm{d}}{\mathrm{d}t} A_i^{\exp(t\theta_i)} = e^{-t\theta_i} \left(A_i\theta_i - \theta_i A_i + \mathrm{d}\theta_i \right) e^{t\theta_i} = e^{-t\theta_i} \mathcal{D}_A \theta_i e^{t\theta_i}, \tag{5.5}$$

which at $t = 0$ gives how Θ acts on the connection:

$$A_i^\Theta = \left. \frac{\mathrm{d}}{\mathrm{d}t} A_i^{\exp(t\theta_i)} \right|_{t=0} = A_i\theta_i - \theta_i A_i + \mathrm{d}\theta_i = \mathcal{D}_A \theta_i.$$

These are the local representatives of $\mathcal{D}_A\Theta \in \Omega^1(M, \operatorname{ad} P)$. □

The following question, then, is what to take as the subspace N in the decomposition $T_y M = T_y(G \cdot y) \oplus T_y N$. Working in the Sobolev completions of the various spaces again, a logical choice would be the orthogonal complement of $\mathscr{G} \cdot A$. Let $\mathcal{D}_A\Theta \in T_A(G \cdot A) = \operatorname{im}(\mathcal{D}_A)$; then we would like to find elements $a \in T_A\mathscr{A}$ such that $0 = (a, \mathcal{D}_A\Theta)$. Connections satisfying $\mathcal{D}_A^\dagger a = 0$ satisfy this condition. Thus $\ker(\mathcal{D}_A^\dagger)$ is our slice, and we have the orthogonal decomposition

$$T_A\mathscr{A} = T_A(\mathscr{G} \cdot A) \oplus \ker(\mathcal{D}_A^\dagger) = \operatorname{im}(\mathcal{D}_A) \oplus \ker(\mathcal{D}_A^\dagger). \tag{5.6}$$

Now, let p_- be the projection onto the anti-self-dual part of a form, and $\mathcal{D}_A^- := p_- \circ \mathcal{D}_A$. Consider the sequence

$$0 \longrightarrow \Omega^0(M, \mathrm{ad}\, P) \xrightarrow{\mathcal{D}_A} \Omega^1(M, \mathrm{ad}\, P) \xrightarrow{\mathcal{D}_A^-} \Omega^2_-(M, \mathrm{ad}\, P) \longrightarrow 0, \tag{5.7}$$

Ω^2_- denoting the space of self-dual 2-forms. A being self-dual, we have $\mathcal{D}_A^- \circ \mathcal{D}_A = F_A^- = 0$, so this sequence forms a complex, called the *deformation complex*. For M compact, this complex is elliptic, so that the cohomology groups H_A^i have finite dimensions. Its index s is on the one hand $s = -\dim H_A^0 + \dim H_A^1 - \dim H_A^2$, while on the other hand it can be calculated using the Atiyah-Singer index theorem [28] giving

$$p_1(\mathrm{ad}\, P) - \tfrac{1}{2} \dim G\, (\chi - \tau) = s = -\dim H_A^0 + \dim H_A^1 - \dim H_A^2. \tag{5.8}$$

where $p_1(\mathrm{ad}\, P)$ is the first Pontryagin class of the adjoint bundle, χ is the Euler characteristic of M and τ is the signature of M.

The middle cohomology group H_A^1 is of particular interest to us: Let A be a self-dual connection, and $\tau \in \Omega^1(M, \mathrm{ad}\, P)$. Then we have for the anti-self-dual part of the curvature of the curve $A_t := A + t\tau$:

$$\begin{aligned} F_t &= \mathrm{d}(A + t\tau) + (A + t\tau) \wedge (A + t\tau) \\ &= \mathrm{d}A + A \wedge A + t\mathrm{d}\tau + t\tau \wedge A + tA \wedge \tau + t^2 \tau \wedge \tau \\ &= F + t\mathrm{d}\tau + t[A, \tau] + t^2 \tau \wedge \tau \\ &= F + t\mathcal{D}_A \tau + \frac{1}{2} t^2 [\tau, \tau]. \end{aligned} \tag{5.9}$$

Thus, A_t is self-dual up to first order in t if and only if $\mathcal{D}_A \tau$ is self-dual, i.e. $\tau \in \ker(\mathcal{D}_A^-)$. Thus, $H_A^1 = \ker(\mathcal{D}_A^-)/\,\mathrm{im}(\mathcal{D}_A)$ consists of the connections satisfying the linearized self-duality equation, modulo the gauge orbit. This is exactly the tangent space $T_A \mathscr{M}$.

Proposition 5.6 *Denote the isotropy group of A by $\Gamma_A := \{g \in \mathscr{G} \mid g(A) = A\}$. Then*

$$\ker(\mathcal{D}_A) = \mathrm{Lie}(\Gamma_A). \tag{5.10}$$

Proof Suppose that $\Theta \in \mathrm{Lie}(\Gamma_A)$, so that $\exp(t\Theta) \in \Gamma_A$ for all t. Then $A_i = A_i^{\exp(t\Theta)}$ is independent of t, so $\mathcal{D}_A \Theta = A_i^\Theta = \frac{\mathrm{d}}{\mathrm{d}t} A_i^{\exp(t\Theta)} \Big|_{t=0} = 0$, so $\Theta \in \ker(\mathcal{D}_A)$. On the other hand, take a $\Theta \in \ker(\mathcal{D}_A)$. Then equation (5.5) shows that $A^{\exp(t\Theta)}$ is independent of t, so that $\exp(t\Theta) \in \Gamma_A$ for all t. This implies $\Theta \in \mathrm{Lie}(\Gamma_A)$. □

Proposition 5.7 *If G is semisimple and A is irreducible then $H_A^0 = 0$.*

Proof Recall that C_A is the centralizer of $\mathrm{Hol}(A)$ in G. Considering $\mathcal{D}_A$ as a map from $\Omega^0(M, \mathrm{ad}\, P)$ to $\Omega^1(M, \mathrm{ad}\, P)$, the cohomology group is

$$\begin{aligned} H_A^0 &= \ker(\mathcal{D}_A) = \mathrm{Lie}(\Gamma_A) \\ &= \mathrm{Lie}(C_A) && \text{(Proposition 2.45)} \\ &= \mathrm{Lie}(Z(G)) && \text{(A is irreducible)} \\ &= Z(\mathfrak{g}). && \text{(Proposition C.1)} \end{aligned}$$

Thus H_A^0 is an abelian ideal of the Lie algebra of G, which is by assumption semisimple, so it is 0. □

Definition 5.8 An irreducible self-dual connection A is said to be *regular* if $H_A^2 = 0$. We denote the moduli space of regular, irreducible connections by $\mathscr{M}'$. If $\mathscr{M} = \mathscr{M}'$ then we say that $\mathscr{M}$ is *regular*.

Theorem 5.9 *The moduli space $\mathscr{M}'$ is a smooth manifold with dimension*

$$s = \dim H^1_A = p_1(\operatorname{ad} P) - \tfrac{1}{2}\dim G\,(\chi - \tau). \tag{5.11}$$

Proposition 5.10 *If M is a compact Riemannian manifold with positive scalar curvature, then $\mathscr{M}$ is regular ([27]).*

Thus the moduli space of $P(S^4, \mathrm{SU}(2))$-bundles is regular, and for these bundles we find

$$\dim \mathscr{M} = 8k - 3, \tag{5.12}$$

where k is the instanton number. In particular, we see that in section 4.2.1 the ansatz (4.19) does not give all instantons, since it only gave $5k$ parameters. Only for $k = 1$ we have found them all. For K3 surfaces, this dimension becomes[1]

$$\dim \mathscr{M} = 8k - 12. \tag{5.13}$$

5.2 Moduli spaces over Kähler surfaces

We have seen in section 2.4 that if the base space is a Kähler surface, then holomorphic vector bundles (or more precisely holomorphic structures on smooth vector bundles) are closely related to anti-self-dual connections. This has consequences when one studies the moduli space. For simplicity we restrict our attention in the remainder of this chapter to bundles $\mathcal{E}$ of rank two, with $\Lambda^2\mathcal{E}$ holomorphically trivial; that is to say, holomorphic $\mathrm{SL}(2,\mathbb{C})$ bundles. The theory applies with minor modifications to more general situations. In order to distinguish between smooth bundles and holomorphic structures on smooth bundles, in this section we shall denote the former with E and the latter with $\mathcal{E}$.

Let M be a compact Kähler surface. Recall that if E is a smooth complex bundle over M having a hermitian metric, then a unitary connection A over it defines a holomorphic structure $\mathcal{E}_A$ on it if and only if F_A is a $(1,1)$-form. Denote the space of connections satisfying this constraint by $\mathscr{A}^{1,1}$. Then it will certainly happen that there are connections $A_1, A_2 \in \mathscr{A}^{1,1}$ which are not gauge equivalent to each other, but which yield isomorphic holomorphic bundles $\mathcal{E}_A$.

The problem is that in this case, we must make a distinction between the gauge group $\mathscr{G}$ of E, and the gauge group of the holomorphic structure $\mathcal{E}$ on E. The former, that is $\mathscr{G}$, is modeled on the structure group $\mathrm{SU}(2)$. This means that elements of $\mathscr{G}$ are not only fiber-preserving and linear on the fibers, but they also preserve the Hermitian connection on E. On the other hand, consider a holomorphic structure $\mathcal{E}$ on E, and let us for the moment forget that E has a Hermitian metric. Then the appropriate concept of the gauge group of $\mathcal{E}$ is the group of fiberwise complex linear and fiber-preserving automorphisms, not necessarily leaving any metric invariant. Let us denote this group for the moment by $\mathscr{G}^{\mathrm{C}}$. In the language of Theorem 1.18, they are local holomorphic functions from M to $\mathrm{SL}(2,\mathbb{C})$. This group contains $\mathscr{G}$ as a subgroup, and since $\mathrm{SL}(2,\mathbb{C})$ is essentially the complexification of $\mathrm{SU}(2)$, $\mathscr{G}^{\mathrm{C}}$ can be thought of as the complexification of $\mathscr{G}$.

Now a unitary connection on E is acted upon by gauge transformations of E, but not yet by gauge transformations of $\mathcal{E}$. However, we can extend the action of $\mathscr{G}$ on $\mathscr{A}$ to $\mathscr{G}^{\mathrm{C}}$ as follows. For any $g \in \mathscr{G}^{\mathrm{C}}$ we put $\tilde{g} = (g^\dagger)^{-1}$ (so $g = \tilde{g}$ precisely when $g \in \mathscr{G}$). Then we define an action of $\mathscr{G}^{\mathrm{C}}$ on $\mathscr{A}$ by

$$\begin{aligned} \overline{\partial}_{g(A)} &= g^{-1} \circ \overline{\partial}_A \circ g, \\ \partial_{g(A)} &= \tilde{g}^{-1} \circ \partial_A \circ \tilde{g}. \end{aligned} \tag{5.14}$$

[1] Actually, in this case there is a complication: the scalar curvature for K3 is not strictly positive but zero, so the above theorem does not hold for K3 surfaces. Indeed, some of the moduli spaces $\mathscr{M}_k$ that we will encounter are not smooth, complication our calculations. However, one can still calculate its dimension, which is still given by this formula.

When $g = \tilde{g}$ this just conjugates the full connection d_A by g^{-1}, so this action agrees with the standard action of $\mathscr{G}$ on $\mathscr{A}$. It preserves $\mathscr{A}^{1,1}$. Moreover, it follows immediately from Theorem 2.62 that the holomorphic bundles $\mathcal{E}_{A_1}$ and $\mathcal{E}_{A_2}$ that are induced by A_1 and A_2 are isomorphic if and only if $A_2 = g(A_1)$ for some $g \in \mathscr{G}^{\mathbb{C}}$. Thus, the 'moduli set' of equivalence classes of holomorphic bundles can be identified with the quotient $\mathscr{A}^{1,1}/\mathscr{G}^{\mathbb{C}}$. Let us write $\mathscr{A}^{1,1}/\mathscr{G}^{\mathbb{C}} =: \mathscr{M}_{\text{bundles}}$.

We would like to find an equivalence between anti-self-dual connections and holomorphic structures on E; that is, between $\mathscr{M}$ and $\mathscr{M}_{\text{bundles}}$. Note that any unitary anti-self-dual connection on E lies in $\mathscr{A}^{1,1}$, and therefore induces a holomorphic structure on E. Since the holomorphic structures induced by two connections which are related to each other by a gauge transformation are isomorphic, we already have one of the directions in our equivalence between $\mathscr{M}$ and $\mathscr{M}_{\text{bundles}}$.

Conversely, any holomorphic structure $\mathcal{E}$ on E induces a connection which necessarily lies in $\mathscr{A}^{1,1}$ (namely the unique one which is compatible with $\mathcal{E}$ and the Hermitian metric), but this connection is not necessarily anti-self-dual. Indeed, we have found in Corollary 2.65 that such a connection is only anti-self-dual if $F_A \perp \omega$. Now suppose that we have a holomorphic structure $\mathcal{E}_A$ on E, such that the field strength F_A of the associated unitary connection A satisfies $F_A \perp \omega$. This condition is preserved by $\mathscr{G}$. However, if $g \in \mathscr{G}^{\mathbb{C}}$, then after some work one may show that

$$F_{g(A)} = g^{-1}F_A g + g^{-1}(\bar{\partial}_A(h\partial_A h^{-1}))g, \tag{5.15}$$

where $h = g^\dagger g$. The first term of this is still perpendicular to ω, but the second not necessarily. Therefore, the condition $F_A \perp \omega$ is not necessarily conserved by $\mathscr{G}^{\mathbb{C}}$, but since A and $g(A)$ are clearly related to each other by a gauge transformation, $\mathcal{E}_A$ and $\mathcal{E}_{g(A)}$ are isomorphic to each other as holomorphic structures.[2] Therefore, given a smooth vector bundle, we do not yet have a clear map from bundle isomorphism classes to anti-self-dual connections.

What we must do, then, is study under what conditions on $\mathcal{E}$ the associated compatible connection is always anti-self-dual. This problem is completely solved by the following theorems ([9, p. 211]).

Let ω be the Kähler form of the compact surface M. For any line bundle L over M we define the *degree* of L to be

$$\deg L := \langle c_1(L) \smile [\omega], [M] \rangle. \tag{5.16}$$

Definition 5.11 A holomorphic $\mathrm{SL}(2,\mathbb{C})$ bundle $\mathcal{E}$ over M is *stable* if for each holomorphic line bundle L over M for which there is a nontrivial holomorphic map $\mathcal{E} \to L$ we have $\deg L > 0$.

Theorem 5.12 *Let E be a* $\mathrm{SU}(2)$ *bundle over a compact Kähler surface. Then the instanton moduli space $\mathscr{M}_E$ is naturally identified with the set of equivalence classes of stable holomorphic* $\mathrm{SL}(2,\mathbb{C})$ *bundles which are topologically equivalent to E.*

Theorem 5.13 *Let V be a* $\mathrm{SO}(3)$ *bundle over a compact, simply connected Kähler surface, such that $w_2(V)$ is the reduction of a $(1,1)$ class c. Then the instanton moduli space $\mathscr{M}_V$ is naturally identified with the set of equivalence classes of stable holomorphic rank-two bundles E with $c_1(E) = c$ and $c_2(E) = \frac{1}{4}(c^2 - p_1(V))$.*

This allows us to use results from the study of moduli spaces of holomorphic vector bundles.

[2] As bundles $\mathcal{E}_A$ and $\mathcal{E}_{g(A)}$ are isomorphic, but they do have different metrics. A and $g(A)$ can both be considered as the unique connection compatible with certain holomorphic structures and metrics, but because $g \in \mathscr{G}^{\mathbb{C}}$, i.e. g does not leave the metric invariant, it follows that the metrics that A and $g(A)$ are compatible with differ from each other. Ultimately, this is the source of the problem: anti-self-duality is a constraint that depends on the metric, so varying the metric changes the concept of anti-self-duality.

Chapter 6

The Path Integral

In this chapter, we shall closely follow [1] to calculate the path integral. Unfortunately, the full physical path integral is too difficult to compute. Therefore, we will calculate the path integral of a topologically twisted theory. This path integral is a topological invariant of the system. When $M = \mathbb{R}^4$or when M is hyper-Kähler, which includes the case $M = \mathrm{K3}$, the theory it describes is actually identical to $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. We will not be able to explain in full detail why it is permissible to use this theory in the case of K3 surfaces, but we will sketch a justification for it.

Before we do so, we discuss some generalities about the path integral. In order to describe the strength of the interactions, one includes a so-called *coupling constant* in it. In the case of Yang-Mills theories, however, it turns out that it is not sufficient to add just this structure constant: we must also include a certain angle, called the vacuum angle. To see this, we must explore the vacuum structure of the theory.

6.1 The vacuum

The following two sections rely on [20, chapter 10 and 12], [29,30] (although it contains some errors), [31], and on [32,33], in which much of the material seems to have been first introduced. Here, we need the structure group to be compact and connected; this allows us to write any element of the structure group as e^x, with $x \in \mathfrak{g}$. As usual, we will take $\mathrm{SU}(2)$.

Now that we have some familiarity with the mathematical background of Yang-Mills theories and instantons, it is time to examine the quantum theory. In particular, we will study the vacuum structure of a quantum Yang-Mills theory on Minkowski spacetime $\mathbb{R}^{3,1}$.

An important concept in Yang-Mills theories is the *path integral*. If $S(x)$ is the action of a theory then it generally looks something like

$$\int [Dx] e^{-iS(x)/\hbar}, \tag{6.1}$$

where the integration is over all paths x. This integration has to be understood in a heuristic sense, because it has not yet been properly mathematically defined. Under a Wick rotation, however, the i vanishes and the integral is well defined.

The integral should be over all physical configurations. Thus, in our case we should integrate over A, but not over all of them: gauge-equivalent gauge potentials determine the same physical configuration. Thus we are allowed to pick a certain gauge. The one that is usually taken is the so-called *temporal gauge*

$$A_0 = 0. \tag{6.2}$$

This gauge is incomplete in the sense that there are still gauge transformations left that respect this gauge. Indeed, if $g : U \to G$ is time-independent, then

$$A_0' = g^1 A_0 g + g^{-1}\partial_0 g = 0.$$

Therefore, we shall consider time-independent gauge transformations from now on. Furthermore, we also restrict our attentions to gauge-transformations that go to the same value at infinity, irrespective of the direction, i.e. $g(|\boldsymbol{x}| \to \infty)$ is uniquely defined. Since any relevant physical interaction should be localized, this should not alter the physical structure of our theory.

Since $g(\boldsymbol{x})$ tends to the same value as $\boldsymbol{x}$ goes to infinity, we may identify all the points at infinity. But $\mathbb{R}^3$ with all the points at infinity identified is topologically the same as the three-sphere S^3, so we should consider gauge transformations $g : S^3 \to \mathrm{SU}(2) \cong S^3$. As we have seen in section 4.2, the homotopy classes of these are determined by their winding number n.[1]

A vacuum then looks like $A = g^{-1}\mathrm{d}g$, with $g : S^3 \to \mathrm{SU}(2)$. Thus, a vacuum $A = g^{-1}\mathrm{d}g$ also has a winding number. It may be calculated explicitly by

$$\frac{1}{24\pi^2}\int \mathrm{Tr}(g^{-1}\mathrm{d}g)^3 = \frac{1}{24\pi^2}\int \mathrm{Tr}(A^3), \tag{6.3}$$

where the powers stand for the wedge product. Note that one can calculate the second integral for any A and any t, but if A is not pure gauge, then the result will in general not be integer, and not independent of t.

Now we make the following definition.

Definition 6.1 Let $g : S^3 \to \mathrm{SU}(2)$ be a gauge transformation. If it is homotopic to the constant gauge transformation, $g \equiv I$, then we call it a *small gauge transformation*. When this is not the case we call it a *large gauge transformation*.

Note that the winding number of a vacuum $A = g^{-1}\mathrm{d}g$ is stable under small gauge transformations, but not under large ones.

Let g be a gauge transformation of the form $g = e^{\omega(x)}$, for $\omega : S^3 \to \mathfrak{su}(2)$. Note that any gauge transformation of this form is small, since $e^{t\omega}$ is a homotopy from it to the constant gauge transformation. It follows from Gauss' law that any physical state should be invariant under such a gauge transformation,

$$e^{\omega}\,|\Psi\rangle = |\Psi\rangle\,.$$

However, there is no such principle for large gauge transformations. On the other hand, any physical observable should be gauge invariant, i.e. its operator should commute with any gauge transformation. For concreteness, let us pick a large gauge transformation, and represent it on the Hilbert space, giving an operator U. Then we may simultaneously diagonalize the Hamiltonian and U. Moreover, U must not change the norm of any physical state, so that its eigenvalues must be of the form $e^{i\theta(U)}$ for some real angle $\theta(U)$.

We establish two immediate facts about this angle $\theta(U)$:

- The angle $\theta(U)$ only depends on the homotopy class of U. Suppose g and g' are two gauge transformations, both having winding number n. Represent them on the Hilbert space by unitary operators U and U' respectively. Since the map which maps a homotopy class of a gauge transformation to its winding number is a homomorphism, $g'g^{-1}$ has winding number $n - n = 0$, i.e. $g'g^{-1}$ is small. Therefore,

$$U'\,|\Psi\rangle = (U'U^{-1})U\,|\Psi\rangle = (U'U^{-1})e^{i\theta(U)}\,|\Psi\rangle = e^{i\theta(U)}\,|\Psi\rangle\,.$$

[1] Even if g is not time-independent, i.e. a smooth map $\mathbb{R} \times S^3 \to S^3$, then it has a winding number which can be found by calculating the winding number of g at any time t. This winding number cannot change as t runs over $\mathbb{R}$, because if it did, then g would be a homotopy from one map $g(t_1, \cdot)$ to another $g(t_2, \cdot)$ which have different winding numbers. This is an impossibility.

- It is a homomorphism. Suppose that U and U' are any two gauge transformations, then

$$e^{i\theta(UU')}\,|\Psi\rangle = UU'\,|\Psi\rangle = e^{i\theta(U)}e^{i\theta(U')}\,|\Psi\rangle = e^{i(\theta(U)+\theta(U'))}\,|\Psi\rangle\,.$$

Let g_1 be a gauge transformation of winding number 1; then we define $\theta := \theta(g_1)$. If g_n is any gauge transformation of winding number n, the above two facts have the consequence that that if U_n represents it on the Hilbert space,

$$U_n\,|\Psi\rangle = e^{in\theta}\,|\Psi\rangle\,. \tag{6.4}$$

Now consider vacuum states, i.e. states of the form $|g^{-1}\mathrm{d}g\rangle$. Since these are topologically classified by the winding number n of g, we write $|n\rangle$. Since these are all gauge equivalent to each other under large gauge transformations, one would expect we can pick any of these as our vacuum state. However, we have already seen that any *physical* state must be an eigenvector of U_1, but these vacua are not. Indeed, write $|n\rangle = |A\rangle$ for some vacuous gauge potential A. Let g_1 be a gauge transformation of winding number 1, and let U_1 act on $|A\rangle$ by $U_1\,|A\rangle = |g_1(A)\rangle$. Then

$$U_1\,|n\rangle = |n+1\rangle\,. \tag{6.5}$$

Therefore, the vacua $|n\rangle$ are no true vacua at all, since they are not even physical states to begin with. A true vacuum state $|\mathrm{vac}\rangle$ must satisfy

$$U_1\,|\mathrm{vac}\rangle = e^{i\theta}\,|\mathrm{vac}\rangle\,. \tag{6.6}$$

This is solved by

$$|\theta\rangle := \sum_{n\in\mathbb{Z}} e^{-in\theta}\,|n\rangle\,; \tag{6.7}$$

indeed, $U_1\,|\theta\rangle = \sum_n e^{-in\theta}\,|n+1\rangle = e^{i\theta}\sum_n e^{-i(n+1)\theta}\,|n+1\rangle = e^{i\theta}\,|\theta\rangle$.

Now, since U_n commutes with any gauge-invariant operator, it acts as a superselection operator that separates the Hilbert space into distinct sectors, labeled by their value of θ. Moreover, θ cannot be changed by any gauge-invariant process. To see this, let O be some gauge invariant operator, so that $O = U_1^{-1}OU_1$. We calculate the expectation value

$$\begin{aligned}\langle\theta|\,O\,|\theta'\rangle &= \langle\theta|\,U_1^{-1}OU_1\,|\theta'\rangle = \langle\theta|\,U_1^{\dagger}OU_1\,|\theta'\rangle = \langle\theta|\,e^{-i\theta}Oe^{i\theta'}\,|\theta'\rangle\\ &= e^{i(\theta'-\theta)}\,\langle\theta|\,O\,|\theta'\rangle\,.\end{aligned}$$

Barring $\langle\theta|\,O\,|\theta'\rangle$ being 0, this can only hold if $\theta = \theta'$, so no physical operator has nonvanishing matrix elements between states labeled by different θ. Moreover, one may write a non-vacuous physical state $|\psi\rangle$ in the form $O_1O_1\cdots O_n\,|\theta\rangle$, where the O_i are gauge-invariant. Therefore, θ is a fixed constant of a quantized Yang-Mills theory; all of nature is in a single, fixed θ-superselection sector.

6.2 The coupling constant

Usually, the Yang-Mills action is defined with a *coupling constant*

$$S_{\mathrm{YM}} = -\frac{1}{g^2}\int_M \mathrm{Tr}(F\wedge\star F) = \frac{1}{g^2}\int_M (|F_+|^2 + |F_-|^2)\,\mathrm{dVol}. \tag{6.8}$$

Classically this makes no difference. In quantum theories, however, one is usually interested in an entity called the *partition function*, *vacuum expectation value* or *path integral* $\langle 0|\,e^{iHT/\hbar}\,|0\rangle$. In our case, this becomes after a Wick rotation

$$Z = \langle\theta|\,e^{-HT/\hbar}\,|\theta\rangle = \sum_{n,n'} e^{i\theta(n-n')}\,\langle n|\,e^{-HT/\hbar}\,|n'\rangle = \sum_{n,q} e^{-i\theta q}\,\langle n|\,e^{-HT/\hbar}\,|n+q\rangle\,,$$

where we set $q := n' - n$. However, the amplitude is independent of n. Indeed, if U_n is a generator of a large gauge transformation of winding number k, then

$$\begin{aligned}\langle n| e^{-HT/\hbar} |n+q\rangle &= \langle 0| U_n^\dagger e^{-HT/\hbar} U_n |q\rangle = \langle 0| e^{-HT/\hbar} U_n^\dagger U_n |q\rangle \\ &= \langle 0| e^{-HT/\hbar} |q\rangle,\end{aligned}$$

so the path integral becomes

$$Z = K \sum_q e^{-i\theta q} \langle 0| e^{-HT/\hbar} |q\rangle = K \sum_q e^{-i\theta q} \int [DA^q] e^{-S_{\mathrm{E}}/\hbar} \tag{6.9}$$

Here, A^q is a connection connecting the vacuum 0 at $t = -\infty$ with one having winding number q at $t = +\infty$. K is a normalization constant encoding the infinity coming from the summation over N. Since in quantum field theoretic calculations one always also divides by Z, it is not physically important.

We can now bring instantons into the picture. We present space as a large cylinder $\mathbb{R} \times S^3$, and suppose that A^q is one of the connections in the path integral (6.9). Thus, at $-\infty$ and ∞ it is pure gauge, with winding number 0 and q respectively. If we calculate c_2 of the bundle on which such a connection lives, we get

$$\begin{aligned}c_2 = \frac{1}{8\pi^2} \int F \wedge F = \int \mathrm{d}\,\mathrm{Tr}(A \wedge F - \tfrac{1}{3} A^3) &= -\frac{1}{24\pi^2} \int_{t=\infty} A^3 + \frac{1}{24\pi^2} \int_{t=-\infty} A^3 \\ = -q,\end{aligned} \tag{6.10}$$

the difference of the winding numbers of A at t_1 and t_2. On the other hand, we recognize this as (a factor times) the action of an instanton with winding number q. Since instantons are the absolute minima of the action, they are the leading contribution in the path integral (6.9). Thus, instantons may be seen as a tunneling process, connecting one vacuum with another.

If we now write q as $q = -c_2 = \frac{-1}{8\pi^2} \int \mathrm{Tr}(F \wedge F)$, then combining the two exponentials in the expression above gives

$$e^{-S_{\mathrm{E}}} e^{-i\theta q} = \exp\left[\int \left(\frac{1}{g^2} \mathrm{Tr}(F \wedge \star F) + \frac{i\theta}{8\pi^2} \mathrm{Tr}(F \wedge F)\right)\right] \tag{6.11}$$

The action is often defined as having this extra θ-term. Since the second term may be written as the total differential of the Chern-Simons form associated to $c_2(F)$, it has no classical effect; alternatively, on extremizing it, it may be seen to yield the Bianchi-identity. Quantummechanically, however, it does have an effect due to winding.

Defining

$$\tau := \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}, \tag{6.12}$$

the action then becomes

$$\begin{aligned}S_{\mathrm{E}} &= -\int \left(\frac{1}{g^2} \mathrm{Tr}(F \wedge \star F) + \frac{i\theta}{8\pi^2} \mathrm{Tr}(F \wedge F)\right) \\ &= \frac{1}{4\pi} \int \left(\frac{4\pi}{g^2}(|F_+|^2 + |F_-|^2) + \frac{i\theta}{2\pi}(|F_+|^2 - |F_-|^2)\right) \mathrm{dVol} \\ &= \frac{1}{4\pi i} \int (-\bar{\tau}|F_+|^2 + \tau|F_-|^2)\, \mathrm{dVol}.\end{aligned} \tag{6.13}$$

With this action, the path integral may be written as

$$Z = K \int [DA] e^{-S_{\mathrm{E}}}. \tag{6.14}$$

where the integral is now no longer limited to connections having a specific value of Q. As an immediate consequence, upon comparing the above formula with the one we started with, $Z = \langle\theta| e^{-HT/\hbar} |\theta\rangle$, we see that the quantum theory of this action has only one ground state, $|\theta\rangle$, instead of infinitely many.

The θ-term has a physically observable effect on the theory. To see this, note that (in Riemannian spaces)

$$F \wedge F = F \wedge \star\star F = \langle F, \star F\rangle \,\mathrm{dVol} = \left(\sum_a F^a_{\mu\nu} \, (\star F)^{a\mu\nu}\right) \mathrm{dVol}$$

If we write F as a matrix,

$$F^a = \begin{pmatrix} 0 & -E^a_1 & -E^a_2 & -E^a_3 \\ E^a_1 & 0 & B^a_3 & -B^a_2 \\ E^a_2 & -B^a_3 & 0 & B^a_1 \\ E^a_3 & B^a_2 & -B^a_1 & 0 \end{pmatrix}, \quad \text{or} \quad \begin{aligned} E^a_i &= F^a_{i0} \\ B^a_i &= \tfrac{1}{2}\epsilon_{ijk}F^a_{jk} \end{aligned} \tag{6.15}$$

then we see that $\sum_a F^a_{\mu\nu} \, (\star F)^{a\mu\nu} = \sum_a \boldsymbol{E}^a \cdot \boldsymbol{B}^a$. Under parity $\boldsymbol{B}$ switches sign while $\boldsymbol{E}$ remains the same, and under time reversal $\boldsymbol{B}$ remains the same while $\boldsymbol{E}$ changes sign. Thus the θ-term breaks P and T-symmetry. It preserves C symmetry, so that it violates CP-symmetry. The strong nuclear force, however, is described by a $\mathrm{SU}(3)$ gauge theory, and it should not have a broken P and CP symmetry. Experimentally the upper bound for θ was found to be $\theta < 10^{-9}$. This strongly suggest that θ should be zero. However, in the current formalism there is no explanation for this. This is known as the strong CP-problem.

6.3 The topologically twisted theory

Here we shall try to give a justification for using the topologically twisted theory as described in [1]. The idea is as follows. Consider an orientable vector bundle $E \to M$, of which the rank equals the dimension of M, which is even, and a generic section s on E. Then it is a well-known result that the Euler characteristic of E may be calculated by counting the zeros of s, with a plus or minus sign depending on whether s preserves or reverses the orientation respectively. Now, if s is a section of which the zeros are not isolated but are submanifolds of M, then what one must count is no longer just ± 1 for each of these submanifolds, but their Euler characteristics. We shall then use this formalism on the section $s(A) = F_+(A)$, i.e. the self-dual part of F. By setting this to zero and somehow dividing out the gauge symmetry, we end up discussing moduli-spaces of instantons, and using the formalism above then results in a path integral which is a sum over the Euler characteristics of the moduli spaces.

As explained, this section serves to give an idea as to how one obtains the expression (6.26) for the partition function, introduced in [1], and to bridge the gap between the chapters above and the calculations that will follow. It will, however, be very brief and sketchy, because treating these matters in full detail is outside of the scope of this thesis.

First, we briefly introduce the Mathai-Quillen formalism, mainly following [34]. Let $E \to M$ be a vector bundle. Then as explained above, one may calculate its Euler characteristic by counting the zeros of a generic section to it,

$$\chi(E) = \sum_{x \in \ker(s)} \nu_s(x), \tag{6.16}$$

where $\nu_s(x)$ is the degree or index of the zero of s at x. On the other hand, we have also seen that the Euler characteristic may be calculating by integrating the Euler class,

$$\chi(E) = \int_M e(E). \tag{6.17}$$

The idea of the Mathai-Quillen formalism is to use another form, depending on both the connection and a section s, which we will denote by $e_s(E)$, such that

$$\chi(E) = \int_M e_s(E). \tag{6.18}$$

If s is the zero-section, then this equation reduces to (6.17), and when s has isolated zeros then it reduces to (6.16). Therefore, this way of evaluating the Euler characteristic may be seen as a continuous path between (6.16) and (6.17).

Suppose E is a vector bundle associated to a principal G-bundle P, $E = P \times_\rho F$. Then one can represent forms on E by basic forms on $P \times F$, as shown in section 2.3:

$$\Omega^*(E) = \Omega^*_G(P \times F), \tag{6.19}$$

and sections of E may be written as G-equivariant maps from P to F. Moreover, via the projection $\pi : P \to M$, E pulls back to the trivial vector bundle $\pi^* E = P \times F$ over P, whose induced connection and curvature we also denote by ∇ and F. Furthermore, we choose a metric on F and denote its coordinates by ξ^a. With this understood, we define the *Thom form*:

$$\Phi(E) = \frac{1}{(2\pi)^{m/2}} \int \mathrm{d}\chi \, \exp(-t^2|\xi|^2/2 + t\nabla\xi^a\chi_a + \chi_a F_{ab}\chi_b/2). \tag{6.20}$$

Here m is the rank of the vector bundle and $t > 0$. Then we have the following.

Proposition 6.2 *The Thom form is closed, integrates to 1, and its derivative with respect to t is exact.*

Now the trick is to pull back the Thom form via a section s of E, giving

$$s^*\Phi(E) = \frac{1}{(2\pi)^{m/2}} \int \mathrm{d}\chi \, \exp(-t^2|s|^2/2 + t\nabla s^a\chi_a + \chi_a F_{ab}\chi_b/2). \tag{6.21}$$

We immediately notice that if s is the zero section, then the first two terms drop out and we are left with an integral which calculates exactly the Pfaffian of F; i.e. in this case $s^*\Phi(E) = e(E)$.

Proposition 6.3 *The cohomology class of $s^*\Phi(E)$ is independent of s and the metric.*

Therefore, $s^*\Phi(E) = e(E)$ for any section s and $t > 0$; in other words, it is a topological invariant. Then we may consider the $t \to \infty$ limit and integrate over M. Assume that s only has isolated and non-degenerate zeros. Then in the limit $t \to \infty$ only the points where s is zero contribute, and at these points, it is possible to choose a coordinate system $\{x^i\}$ on M such that s is at most linear in each of the x^i. Then the first term leads to a Gaussian integral, the second to one which is linear in the anticommuting variable χ_a, while the third can be transformed away by the scaling $\chi \mapsto \chi/\sqrt{t}$. One finally obtains

$$s^*\Phi(E) = \sum_{x \in \ker(s)} \epsilon_x := \sum_{x \in \ker(s)} \operatorname{sgn}\left(\det\left(\frac{\partial s^a(x)}{\partial x^a}\right)\right). \tag{6.22}$$

Thus this results in a proof of the Gauss-Bonnet theorem.

The situation that will be of most interest to us, however, is a hybrid between such a generic s and the zero section. Suppose that $s = 0$ on a union of submanifolds M_i of M. In equation (6.21) we do the integration over each of the directions separately. Fixing a particular M_i, the integration of the directions normal to M_i proceeds precisely as in the derivation of (6.22), giving the factors ϵ_i. On the other hand, the integration along the directions tangent to M_i, the terms containing s vanish, again leaving only the one involving F. Thus this leads to a factor $\chi(M_i)$. Putting it together, we obtain

$$s^*\Phi(E) = \sum_i \epsilon_i \chi(M_i). \tag{6.23}$$

When one wishes to interpret this quantity as the partition function of a theory, the presence of these signs are inappropriate. However, if one simply removes them from the sum then the result will generally no longer be a topological invariant of the system, removing much of its mathematical use and interest. Vafa and Witten solved this problem by adding certain auxiliary degrees of freedom, altering the system. When done in the appropriate way, all the signs ϵ become 1, while the number of zeros of the new section s remains the same as that of the unaltered theory. In this way, the signs vanish from the formulae, while the calculated entity is still a topological invariant (although of a modified system). Essentially, this is the topological twist.

To apply this formalism to the situation we wish to consider, one takes as base space the space of connections, and as bundle E the space of self-dual two-forms with values in the adjoint representation. As section one takes

$$s(A) = F^{+}(A), \tag{6.24}$$

the self-dual part of A – i.e. the section of which the zeros are instantons. Some careful and complicated reasoning then results in a path integral of the form

$$Z(\tau) = \frac{1}{\#Z(G)} \sum_{n} q^{n} \chi(\mathscr{M}_n), \tag{6.25}$$

where $\mathscr{M}_n$ is the moduli space of n-instantons and $q = e^{2i\pi\tau}$. When considering the SO(3) theory, one must also sum over w_2. This argument relies on two critical issues:

- Parallel to the above discussion, the system one considers has to be such that it is possible to add extra degrees of freedom, in order to remove the signs while keeping the partition function a topological invariant. Fortunately, when M is Kähler with $R \geq 0$ and when the gauge group is locally a product of SU(2)'s, this turns out to be possible. This includes our case of $M = \mathrm{K3}$ and $G = \mathrm{SU}(2)$.
- The Euler characteristic $\chi(\mathscr{M}_n)$ originally entered our arguments as an integral over the Euler class, but $\mathscr{M}_n$ is not compact because instantons can shrink to zero size. Being a topological space, and in fact a manifold, it obviously still has an Euler characteristic; we just have to calculate it in a different way. We shall address this problem later.

Twisting on Calabi-Yau spaces

There is a more algebraic description of topological twisting, which shows more clearly why we chose K3 surfaces as our base space. The structure group of a general differential orientable 4-manifold is SO(4), which is locally isomorphic to $K := \mathrm{SU}(2)_{\mathrm{L}} \otimes \mathrm{SU}(2)_{\mathrm{R}}$. Furthermore, in $\mathcal{N} = 4$ supersymmetric theory there is the R-symmetry group SU(4). Therefore, the full symmetry group of the theory is $\mathrm{SU}(2)_{\mathrm{L}} \otimes \mathrm{SU}(2)_{\mathrm{R}} \otimes \mathrm{SU}(4)$. The idea of twisting is to take the right $\mathrm{SU}(2)_{\mathrm{R}}$, and replacing it by a diagonal combination of $\mathrm{SU}(2)_{\mathrm{R}} \otimes \mathrm{SU}(4)$. The result is a new rotation group $K' = \mathrm{SU}(2)_{\mathrm{L}} \otimes \mathrm{SU}(2)'_{\mathrm{R}}$, embedded in the full symmetry group, which we consider to be the rotation group of a new theory. This is the twisted theory. If we take a base space of which the structure group can be reduced to the left $\mathrm{SU}(2)_{\mathrm{L}}$, then the result is a theory which is insensitive to the twisting. That is, on Calabi-Yau manifolds the twisted theory coincides with the untwisted one. The simplest example would therefore be compact simply connected Calabi-Yau 2-folds; i.e. K3 surfaces.

6.4 S-duality

When considering the Maxwell equations, the only thing that keeps them from being symmetric under exchanging electricity and magnetism with each other is the absence of magnetic monopoles. In any quantum theory the magnetic charge of such a monopole, g_m, has to obey the Dirac quantization condition, which in suitable units can be written as $g_m = 4\pi n/e$. Here $n \in \mathbb{Z}$.

An extension of this symmetry, originally proposed by Montonen and Olive [35], is the conjecture that $\mathcal{N} = 4$ supersymmetric Yang-Mills theories are invariant under a similar symmetry. This symmetry is called *S-duality*, and it sends τ to $-1/\tau$ and the gauge group G to the dual group $\widehat{G}$, which is the unique group whose weight lattice is the dual of that of G. If $G = \mathrm{SU}(N)$, then $\widehat{G} = \mathrm{SU}(N)/\mathbb{Z}_n$. Thus when $N = 2$, then $\widehat{G} = \mathrm{SO}(3)$. Concretely, then, one would expect the partition functions to transform as follows:

$$Z_{\mathrm{SU}(2)}(-\tau^{-1}) = Z_{\mathrm{SO}(3)}(\tau).$$

However, two natural generalizations of this come to mind. One is the possibility that Z might transform like a *modular form*:

$$Z_{\mathrm{SU}(2)}(-\tau^{-1}) = (-i\tau)^{w/2} Z_{\mathrm{SO}(3)}(\tau),$$

for some w. Another comes from the fact that the leading power of q in a modular object is not always 0. For example, the Dedekind eta function which will play a central role in the calculations to come has an integral expansion multiplied by $q^{1/24}$. As a result of this, the partition function should be modified by an overall multiplicative factor to be

$$Z_G(\tau) = \frac{q^{-s}}{\#Z(G)} \sum_n q^n \chi(\mathscr{M}_n), \tag{6.26}$$

for some s.

We have seen that SO(3)-bundles are classified by their second Stiefel-Whitney class w_2 and the instanton number $n = -\frac{1}{4}p_1$. In this case, the sum in the expression above should run not only over n but also over w_2. Let us write for now $v := w_2$ and Z_v for each of the partition functions with fixed v. Vafa-Witten showed in their article that if the theory is to be S-dual, then in the case of SU(2) and SO(3) the Z_v must transform as

$$Z_v(-\tau^{-1}) = 2^{-b_2(M)/2}(-i\tau)^{w/2} \sum_u (-1)^{Q(u,v)} Z_u(\tau). \tag{6.27}$$

Moreover, the partition function for the SU(2) theory is the same as the contribution of the partition function with zero v, times a factor:

$$Z_{\mathrm{SU}(2)}(\tau) = 2^{b_1(M)-1} Z_0(\tau). \tag{6.28}$$

Combining the last two formulae shows what the SU(2) partition function should transform to under S-duality:

$$\begin{aligned} Z_{\mathrm{SU}(2)}(-\tau^{-1}) &= 2^{-1+b_1(M)} Z_0(-\tau^{-1}) \\ &= 2^{-1+b_1(M)-b_2(M)/2}(-i\tau)^{w/2} \sum_u (-1)^{Q(u,0)} Z_u(\tau) \\ &= 2^{-\chi(M)/2}(-i\tau)^{w/2} Z_{\mathrm{SO}(3)}(\tau). \end{aligned} \tag{6.29}$$

In the remainder, we shall take as partition function (6.26), and as S-duality equations (6.27) and (6.29).

Vafa and Witten were among the first to clearly formulate S-duality in the context of $\mathcal{N} = 4$ supersymmetric Yang-Mills theories, and test a number of examples for it. They have not rigorously showed it in the case which is of most interest to us, which is SU(2), SO(3) and K3 surfaces; instead, they derived the partition functions of these two theories almost completely via other means and calculated the last unknown by assuming S-duality. Since that time S-duality has been shown to hold in most of these theories, in more general settings. Therefore, we shall assume that S-duality holds, and use it to derive a part of the partition function like Vafa and Witten.

6.5 The moduli space as symmetric products

Suppose X is a set. Then we denote with $S(X)$ its *symmetric group*, i.e. the group consisting of all bijections. If $X = \{1, \ldots, n\}$ then we write S_n for its symmetric group. Now we let it act on a product of smooth manifolds by permuting the factors:

$$M^{(n)} := (M \times \cdots \times M)/S_n. \tag{6.30}$$

This space is an orbifold, called the *symmetric product*, and it has singularities corresponding to the fixed points of the group action. These singularities may be resolved, giving a new smooth space $M^{[n]} := \widetilde{M^{(n)}}$.

Let $M = \mathrm{K3}$. If E is an $\mathrm{SU}(2)$ bundle on K3 with instanton number[2] k, then one can seek a convenient description of E by finding a line bundle L on K3 such that the index of the $\overline{\partial}$ operator of $L^{-1} \otimes E$ is 1 [36–38]. It turns out that in this case $H^0(\mathrm{K3}, L^{-1} \otimes E)$ consists of a single holomorphic section (up to scalar multiples). The number of zeros of s, counted with multiplicity, equals the second Chern class of $L^{-1} \otimes E$, which turns out to be $2k - 3$. Thus, if L exists, one has a natural way to extract from E a configuration of $2k - 3$ points on K3. Conversely, when an appropriate L exists, given a configuration of $2k - 3$ points one can construct a unique E.

Recalling Theorem 5.12, this suggests that when such an L exists, the instanton moduli space can be identified with the space of configurations of $2k - 3$ distinct but unordered points – that is, $\mathrm{K3}^{(2k-3)}$. Resolving the singularities, we obtain the identification $\mathscr{M}_k \approx \mathrm{K3}^{[2k-3]}$ as the algebrogeometric compactification of the instanton moduli space. This compactification has the same Betti numbers and Euler characteristic.

The critical question is then when a suitable L exists. For $\mathrm{SU}(2)$, this turns out to be the case if k is odd, and if a suitable complex structure is chosen on K3. For different values of k, one needs different complex structures on K3, but since the partition function does not depend on the complex structure of the base space, this does not concern us. For $\mathrm{SO}(3)$-bundles with nonzero w_2 an appropriate L always exists. (The instanton k might be a half-integer, but $2k - 3$ is always an integer, so that $\mathrm{K3}^{[2k-3]}$ still makes sense.) Since an $\mathrm{SU}(2)$ theory may also be seen as a $\mathrm{SO}(3)$ theory which has $w_2 = 0$, these two can be unified in demanding either $w_2 \neq 0$ or k is odd. The unknown contribution of the even k's is fully determined by the demand that the $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$-partition functions be S-dual, as we will show below.

As the first step, then, we should calculate $\chi(\mathrm{K3}^{[n]})$. We shall see that we can do this in terms of certain Euler characteristics coming from the blow-down $\mathrm{K3}^{(n)}$. To calculate these Euler characteristics, we first combine all the symmetric products $\mathrm{K3}^{(n)}$ into a larger space, using the factor q^n in equation (6.26) as a formal parameter to create a disjoint union – i.e. to keep track of n. We then calculate the Euler characteristic of this larger space. The result will be a sum over the Euler characteristics of all $\mathrm{K3}^{(n)}$ spaces multiplied by q^n; that is, it will automatically have the form of the sum in equation (6.26). References for this approach are [39,40].

Definition 6.4 If $V = V^+ \oplus V^-$ is a graded vector space with even part V^+ and odd part V^-, then we define the *superdimension* of V as $\operatorname{sdim} V = \dim V^+ - \dim V^-$.

Note that using this notation, the Euler characteristic of a space is just the superdimension of its cohomology, $\chi(M) = \operatorname{sdim} H^*(M)$.

Definition 6.5 If M is a topological space, then we define

$$S_q M := \coprod_{n \geq 0} q^n M^{(n)}. \tag{6.31}$$

[2] At this point, we switch notation. Henceforth, k shall denote the instanton number, while $n = 2k + 3$ will be the power of the corresponding symmetric product, to be introduced shortly.

If V is an even vector space, then we modify this to be compatible with the vector space structure

$$S_q V := \bigoplus_{n\geq 0} q^n V^{\otimes n}/S_n, \tag{6.32}$$

and similarly, if V is an odd vector space, then

$$S_q V := \bigoplus_{n\geq 0} q^n (\bigwedge^n V)/S_n, \tag{6.33}$$

It is not hard to see that for even respectively odd vector spaces,

$$\dim S_q V^+ = \sum_{n\geq 0} q^n \binom{n+\dim V^+ - 1}{n} = (1-q)^{-\dim V^+},$$
$$\dim S_q V^- = (1-q)^{\dim V^-}.$$

These two can be combined in the single formula which holds for arbitrary graded vector spaces

$$\operatorname{sdim} S_q V = (1-q)^{-\operatorname{sdim} V}. \tag{6.34}$$

6.5.1 The orbifold Euler characteristic

Now we shall consider the Euler characteristics of orbifolds, and in particular, of symmetric products. Let G be a finite group acting on a compact differentiable manifold M. Then the quotient space M/G will in general not be a smooth manifold; instead it is an orbifold. Topological invariants like the Betti numbers for such a space are well-known.

Proposition 6.6 *We have for the Betti numbers of the orbifold M/G*

$$b_i(M/G) = \dim H^i(M)^G = \frac{1}{|G|}\sum_{g\in G} \operatorname{Tr}_{H^i(M)}(g^*), \tag{6.35}$$

where $H^i(M)^G$ is the part of $H^i(M)$ which is left invariant under G.

Thus for the topological Euler characteristic we get

$$\chi_{\text{top}}(M/G) = \operatorname{sdim} H^*(M)^G. \tag{6.36}$$

By the Lefschetz fixed point formula we have

Proposition 6.7 *$\chi(M/G)$ is determined by the fixed point sets M^g as follows*

$$\chi_{\text{top}}(M/G) = \frac{1}{|G|}\sum_{g\in G} \chi_{\text{top}}(M^g). \tag{6.37}$$

Thus, this allows us to calculate the Euler characteristics of a quotient by examining its fixed points M^g. However, suppose that $x \in M^g$ and $h \in C_g$, i.e. $hg = gh$. Then clearly $g(hx) = h(gx) = hx$; that is, $hx \in M^g$. Thus the centralizer C_g still acts on the fixed-point space M_g. To include the additional information that this action contains, we make the following definition.

Definition 6.8 The *orbifold Euler characteristic* for an orbifold M/G is defined by

$$\chi_{\text{orb}}(M/G) = \sum_{[g]} \chi_{\text{top}}(M^g/C_g). \tag{6.38}$$

Here the sum runs over the conjugacy classes $[g]$ of G.

If e is the identity, then $M^e/C_e = M/G$. Thus the topological point of view only examines the trivial class $g = e$, while $\chi_{\text{orb}}(M/G)$ takes into account more of the structure of the action of G on M.

By using the Lefschetz formula again, we obtain the following.

Proposition 6.9 *We have*

$$\chi_{\text{orb}}(M/G) = \frac{1}{|G|}\sum_{gh=hg}\chi_{\text{top}}(M^{g,h}) = \frac{1}{|G|}\sum_{gh=hg}\operatorname{sdim} H^*(M^{g,h}), \tag{6.39}$$

where $M^{g,h}$ is the set left invariant by both g and h.

The reason as to why this orbifold Euler characteristic is important to us, is that in certain cases, the *orbifold* Euler characteristic of M/G equals the *topological* Euler characteristic of its resolution $\widetilde{M/G}$. Indeed, this holds for Kummer surfaces. Let T be a torus, and recall that σ is the involution on it sending x to $-x$. Then it is fairly easy to see that by both equations (6.38) and (6.39) we indeed have $\chi_{\text{orb}}(T/\{\text{id}, \sigma\}) = 24 = \chi_{\text{top}}(\text{K3})$. Furthermore [40],

Proposition 6.10 *If S is a smooth (complex-)algebraic surface, then we have for the symmetric product and its resolution*

$$\chi_{\text{orb}}(S^{(n)}) = \chi_{\text{top}}(\widetilde{S^{(n)}}) = \chi_{\text{top}}(S^{[n]}). \tag{6.40}$$

6.5.2 The Euler characteristic of symmetric products

Now that we know how to calculate the relevant Euler characteristics in terms of the group action, it is time to determine the properties of the action of the symmetric group on M^n, i.e. the fixed points and conjugacy classes of the symmetric group S_n.

Definition 6.11 A *partition* of a positive integer n is a list of positive integers $\{n_i\}_i$ such that $\sum_i in_i = n$, i.e. it is a (non-unique) way of decomposing n in a sum of lesser integers. We denote the set of all partitions of n by $\mathcal{P}_n$ and the union of all of these by $\mathcal{P} := \cup_n \mathcal{P}_n$.

Each element $\sigma \in S_n$ may be written as a product of disjoint cycles of length i; suppose that there are n_i cycles in σ of length i. If σ leaves a particular number invariant, then we include this in the decomposition of σ into cycles by writing it as a one-cycle. In this way, σ determines a set of nonnegative integers $\{n_i\}$ such that $\sum_i in_i = n$. We call this its *cycle type*. We see that each such a cycle type is a partition of n. For example, $(1\,3\,5)(2\,7\,6)(4) \in S_7$ has cycle type $(1, 0, 2, 0, 0, 0, 0)$, and $\sum_i in_i = 1\cdot 1 + 3 \cdot 2 = 7$.

Proposition 6.12 *The conjugacy class $[\sigma]$ of $\sigma \in S_n$ is determined by its cycle type, i.e. τ is a conjugate of σ if and only if it has the same cycle type.*

To find out what the centralizer of an element of S_n looks like, we consider the following cases.

- Suppose σ is a cycle of length i. Suppose $\tau \in S_n$ is of the form $\pi\sigma^k$, where k is a nonnegative integer and where π only acts on points that σ leaves invariant. Then τ commutes with σ, so $C_\sigma \supset S(\text{Fix}(\sigma)) \times \langle\sigma\rangle \cong S_{\#\text{Fix}(\sigma)} \times \mathbb{Z}_i$. By computing the order of both sides of this inclusion and using the elementary fact $|[\sigma]| = |G|/|C_\sigma|$, it is possible to prove that this inclusion is actually an equality.
- Suppose $\sigma = \sigma_1 \cdots \sigma_k$ is a product of cycles which are all of length i. Then C_σ obviously contains the intersection of the centralizers C_{σ_k}, i.e. $C_\sigma \supset S(\text{Fix}(\sigma)) \times \langle\sigma_1\rangle \times \cdots \times \langle\sigma_k\rangle \cong S_{\#\text{Fix}(\sigma)} \times \mathbb{Z}_i^k$.
  However, suppose that $\tau \in S_n$ is such that it switches the moved points of σ_{i_1} with those of σ_{i_2}. Then it is not hard to see that τ commutes with $\sigma_{i_1}\sigma_{i_2}$. Furthermore, if we write τ as a product of cycles then it contains only numbers that σ_{i_1} and σ_{i_2} contains, so that τ also commutes with any other σ_i. Therefore, $\tau \in C_\sigma$. It is also easy to see that the group generated by these transformations is isomorphic to S_k.

By using $|[\sigma]| = |G|/|C_\sigma|$ again we can see that these are all the elements of C_σ. Moreover, $S_{\#\,\mathrm{Fix}(\sigma)} \times \mathbb{Z}_i^k$ is normal within C_σ, so we find $C_\sigma = S_{\#\,\mathrm{Fix}(\sigma)} \times \mathbb{Z}_i^k \rtimes S_k$.

Now take a general $\sigma \in S_n$. Then the number of points that it leaves fixed is precisely the number of one-cycles in σ, i.e. n_1. Thus the two arguments above are a sketch of the following.

Proposition 6.13 *If $\sigma \in S_n$ has cycle type $\{n_i\}_{i=1}^n$ then*

$$C_\sigma \cong S_{n_1} \times (S_{n_2} \ltimes \mathbb{Z}_2^{n_2}) \times \cdots \times (S_{n_n} \ltimes \mathbb{Z}_k^{n_n}). \tag{6.41}$$

Now according to equation (6.38), we need to find $(M^n)^\sigma / C_\sigma$.

If $\sigma \in S_n$ is a cycle of length i, then it leaves an element in M^n fixed if and only if the i points on which it acts coincide. Thus only one of these points can take on an arbitrary values, determining all the others. If σ is an arbitrary element of S_n having cycle type $\{n_i\}$ and acting on $x = (x_1, \ldots, x_n) \in M^n$, then, for each of the cycles in σ there is precisely one element of the n-tuple x that can take on arbitrary values. That is,

$$(M^n)^\sigma \cong \prod_{i>0} M^{n_i}. \tag{6.42}$$

Now we let C_σ act on this. In fact, only the subfactors S_{n_i} in equation (6.41) act nontrivially, giving

$$(M^n)^\sigma / C_\sigma \cong \left(\prod_{i>0} M^{n_i} \right) / C_\sigma \cong \prod_{i>0} (M^{n_i} / S_{n_i}) = \prod_{i>0} M^{(n_i)}. \tag{6.43}$$

We are now finally able to calculate the Euler characteristic of a symmetric product of a space M.

Proposition 6.14 *We have*

$$\chi_{\mathrm{orb}}(S_q M) = \prod_{i>0} (1 - q^i)^{-\chi(M)}. \tag{6.44}$$

Proof First note that the topological number is just [41]

$$\begin{aligned}
\chi_{\mathrm{top}}(S_q M) &= \sum_{n \geq 0} q^n \chi_{\mathrm{top}}(M^{(n)}) = \sum_{n \geq 0} q^n \,\mathrm{sdim}\, H^*(M)^{S_n} = \mathrm{sdim}\, S_q H^*(M) \\
&= (1 - q)^{-\chi(M)}.
\end{aligned}$$

Then

$$\begin{aligned}
\chi_{\mathrm{orb}}(S_q M) &= \sum_{n \geq 0} \sum_{[\sigma] \subset S_n} q^n \chi_{\mathrm{top}}((M^n)^\sigma / C_\sigma) \\
&= \sum_{n \geq 0} \sum_{\{n_j\}_j \in \mathcal{P}_n} q^n \left(\prod_{i>0} \chi_{\mathrm{top}}(M^{(n_i)}) \right) \\
&= \sum_{\{n_j\}_j \in \mathcal{P}} \prod_{i>0} q^{i n_i} \chi_{\mathrm{top}}(M^{(n_i)}) \\
&= \prod_{i>0} \sum_{n_i > 0} q^{i n_i} \chi_{\mathrm{top}}(M^{(n_i)}) \\
&= \prod_{i>0} \chi_{\mathrm{top}}(S_{q^i} M) \\
&= \prod_{i>0} (1 - q^i)^{-\chi(M)}.
\end{aligned}$$

□

This result is closely related to the following function, which we will deal with frequently below.

Definition 6.15 For τ in the upper half complex plane, let $q = e^{2i\pi\tau}$. The *Dedekind eta function* and the function $G(q)$ are defined respectively by

$$\begin{aligned} \eta(\tau) &:= q^{\frac{1}{24}} \prod_i (1-q^i), \\ G(q) &= \eta(\tau)^{-\chi(M)} = q^{-\frac{\chi(M)}{24}} \prod_i (1-q^i)^{-\chi(M)}. \end{aligned} \tag{6.45}$$

In the remainder, we will frequently use both q and τ as arguments for G. These are understood to mean the same, i.e. $G(q) = G(e^{2i\pi\tau})$ and $G(\tau)$ mean the same thing.

The following fact comes from number theory (see for example [42]), and will be very important to us.

Proposition 6.16 *The Dedekind eta function is a modular form of weight $\frac{1}{2}$, that is*

$$\begin{aligned} &\eta(-\tau^{-1}) = \sqrt{-i\tau}\eta(\tau), \quad \textit{and consequently} \\ &G(-\tau^{-1}) = (-i\tau)^{\chi(M)/2} G(\tau). \end{aligned} \tag{6.46}$$

Proposition 6.17 *Under $\tau \to -\tau^{-1}$, G satisfies*

$$\begin{aligned} &G(q^{1/2}) \to 2^{-12}(-i\tau)^{-12} G(q^2), \\ &G(q^2) \to 2^{12}(-i\tau)^{-12} G(q^{1/2}), \\ &G(-q^{-1/2}) \to (-i\tau)^{-12} G(-q^{1/2}). \end{aligned} \tag{6.47}$$

Proof We calculate

$$\begin{aligned} G(q^{1/2}) &= G(e^{2i\pi(\tau/2)}) \to G(e^{2i\pi(-\tau^{-1}/2)}) = G(e^{-2i\pi(2\tau)^{-1}}) \\ &= 2^{-12}(-i\tau)^{-12} G(q^2). \end{aligned}$$

The other identities are proved in a similar fashion. □

Therefore, using these functions, we find the following concise result:

$$\begin{aligned} \sum_n q^n \chi_{\text{top}}(\text{K3}^{[n]}) &= \chi_{\text{top}}\big(\coprod_n q^n \text{K3}^{[n]}\big) = \chi_{\text{orb}}(S_q(\text{K3})) \\ &= \prod_{i>0} (1-q^i)^{-24} = q\eta(\tau)^{-24} = qG(q). \end{aligned} \tag{6.48}$$

6.6 The partition function

In this section we shall finally determine the partition functions $Z_{\mathrm{SU}(2)}(\tau)$ and $Z_{\mathrm{SO}(3)}(\tau)$. It is easiest to first consider SO(3) bundles with nonzero w_2, because then the description of the moduli space as a symmetric product works for all instanton numbers k. For SO(3) theories, the sum in equation (6.26) runs not only over k but also over w_2. Since $H^2(K3, \mathbb{Z})$ is 22-dimensional and torsion-free, w_2 can take 2^{22} values. Fortunately, there is no need to study all of them separately.

Note that since $w_2^2 = p_1 \bmod 4$, by Corollary 2.101 the only possible values that w_2^2 can take is 0 and 2.

Definition 6.18 If $w_2^2 = 0 \bmod 4$ (but w_2 is not zero) then we say that w_2 is of *even type*. If $w_2^2 = 2 \bmod 4$ then we say that w_2 is of *odd type*.

Theorem 6.19 *If two nonzero w_2, w_2' are of the same type, then they have identical moduli spaces.*

Proof If L is some integer lattice, then we say an element $v \in L$ is *primitive* if it cannot be written as nv' for some $n \neq \pm 1$. For any nonzero $v' \in L$, then, there is a primitive v and $n \in \mathbb{Z}$ such that $v' = nv$. Then it is well-known that the orthogonal group of an even unimodular quadratic form acts transitively on primitive vectors of given length [43], if $|\tau| \leq r - 4$, where r is the rank of the form. This is indeed the case for the intersection form of K3. Moreover, for any nonzero w_2 there is at least one lift $w \in H^2(\mathrm{K3}, \mathbb{Z})$ which is primitive. Indeed, take a lift w'. If it is not primitive, then it may be written as $(2n+1)w$ for some n and primitive w. The factor has to be odd, because if it were even then the result would be zero modulo 2. Thus $w_2 = w' \bmod 2 = (2nw + w) \bmod 2 = w \bmod 2$.

Now let $E \to S'$ be an SO(3)-bundle for some K3 surface S', and let f be a diffeomorphism from some other K3 surface S to S'. Then by Theorem 3.11, w_2^2 is necessarily left invariant under such a diffeomorphism f. Moreover, any diffeomorphism of K3 comes from an element $O_Q^+(3, 19; \mathbb{Z})$, those orthogonal transformations that leave the orientation of the positive three-dimensional subspace of $H^2(\mathrm{K3}, \mathbb{Z})$ fixed.

Thus, pick a bundle E over a K3 surface S' with a $w_2(E) =: w'$ of a particular type. Pick any other w of the same type. Then there exists a diffeomorphism $S \to S'$, where S is another K3 surface, such that $f^*(w') = w$. Moreover, consider the pullback bundle f^*E over S. Then by naturality of w_2, we have

$$w_2(f^*E) = f^*(w_2(E')) = f^*(w') = w.$$

A connection A on E induces a connection f^*A on f^*E. Under a gauge transformation g, we would obtain $g^{-1}F_{f^*A}g = g^{-1}f^*(F_A)g$. But the conjugation by g^{-1} only works on the Lie algebra component of F, while the pullback f^* only works on the differential form-component of F. Therefore, F_A is anti-self-dual if and only if F_{f^*A} is too. This implies that the moduli spaces $\mathscr{M}_{w_2,k}$ and $\mathscr{M}_{w_2',k}$ are the same. □

Picking *any* even w_2' and writing $Z_{\text{even}} := Z_{w_2'}$, then, the conclusion is that $Z_{w_2} = Z_{\text{even}}$ for any other even w_2. Via the same argument there is also a Z_{odd}. Thus there are only three partition functions that we must study: two for both of these cases and one for the case $w_2 = 0$. We shall call them Z_{even}, Z_{odd} and Z_0 respectively. We shall call the number of times that these partition functions occur in the sum n_{even} n_{odd} and $n_0 = 1$.

We write the partition function (6.26) as

$$\begin{aligned} Z_{\mathrm{SO}(3)}(\tau) &= q^{-s}\sum_k q^k \chi(\mathscr{M}_{0,k}) + q^{-s}\sum_{w_2 \text{ even}; k} q^k \chi(\mathscr{M}_{w_2,k}) + q^{-s}\sum_{w_2 \text{ odd}; k} q^k \chi(\mathscr{M}_{w_2,k}) \\ &= Z_0(\tau) + n_{\text{even}} Z_{\text{even}}(\tau) + n_{\text{odd}} Z_{\text{odd}}(\tau). \end{aligned} \tag{6.49}$$

Moreover, since a SU(2) bundle is just a SO(3) bundle with zero w_2, we also find

$$Z_{\mathrm{SU}(2)}(\tau) = \frac{1}{2} Z_0(\tau). \tag{6.50}$$

Here, the factor $1/2$ comes from the factor $1/\#Z(G)$ in equation (6.26).

Note that w_2 is odd iff $w_2^2 = 2 \bmod 4 = p_1 \bmod 4$. Since $k = -\frac{1}{4}p_1$, this is the case iff k is a half-integer (i.e. of the form $k' + 1/2$ for some integer k'), which is the case iff $n = 2k - 3$ is even. Similarly, w_2 is even if and only if $n = 2k - 3$ is odd.

6.6.1 Determining n_0, n_{even} and n_{odd}

We should calculate the values for these n's. Recall that the intersection form of K3 is of the form $2(-E_8) \oplus 3H$, where E_8 is the root lattice of the corresponding Lie algebra, and H is the matrix with 1's on its antidiagonal. Modulo 2 one has $2(-E_8) = E_8 \oplus -E_8$, which by a suitable basis transformation may be seen to equal $8H$. Therefore, it suffices to take as intersection form $11H$. This makes the combinatorics of counting the different kinds of bundles straightforward.

Write an element $v \in H^2(\mathrm{K3}, \mathbb{Z}_2)$ as $v = \sum_{i=1}^{22} v_i e_i$, where e_i are generators of $H^2(\mathrm{K3}, \mathbb{Z}) \approx \mathbb{Z}^{22}$, and $v_i = \pm 1$. Write $V_i := v_{2i-1}e_{2i-1} + v_{2i}e_{2i}$. The fact that the intersection form may be written as $11H$ then gives

$$v^2 = \sum_{i=1}^{11} V_i^2, \tag{6.51}$$

and

$$V_i^2 = V_i^\top H V_i = \begin{pmatrix} v_{2i-1} & v_{2i} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v_{2i-1} \\ v_{2i} \end{pmatrix} = 2v_{2i-1}v_{2i}. \tag{6.52}$$

Therefore, $V_i^2 = 2 \bmod 4$ if and only if $v_{2i-1} = v_{2i} = 1$. If either or both of the coefficients v_{2i-1} and v_{2i} is zero, then $V_i^2 = 0 \bmod 4$.

Let us determine n_{even}. v^2 is even if and only if the number of odd V_i's in the sum above is even. For example, if $\sum_{i=1}^{11} V_i^2$ consists of 4 V_i's which are odd, then v^2 will be even. The number of odd V_i's must thus be a multiple of two. As for the other V_i's, they must be even, that is, each of these V_i's must be of the form 0, $v_{2i-1}e_{2i-1}$ or $v_{2i}e_{2i}$. Each of these thus has three ways in which it can be even. Putting things together one obtains

$$\begin{aligned} n_{\text{even}} &= \#\text{ of nonzero } v \text{ such that } v^2 = 0 \bmod 4 \\ &= \sum_{i=0}^{5} \binom{11}{2i} 3^{11-2i} - 1 \\ &= 2^{21} + 2^{10} - 1. \end{aligned} \tag{6.53}$$

Here we subtracted 1 to exclude $v = 0$ which is a separate case (corresponding to n_0). One could obtain n_{odd} via similar arguments, or by just noticing that $n_0 + n_{\text{even}} + n_{\text{odd}} = 2^{22}$. Thus the results are

$$\begin{aligned} n_0 &= 1, \\ n_{\text{even}} &= 2^{21} + 2^{10} - 1, \\ n_{\text{odd}} &= 2^{21} - 2^{10}. \end{aligned} \tag{6.54}$$

6.6.2 Determining the even and odd partition functions

We are now ready to determine these partition functions. In the case of Z_{odd}, w_2 being odd implies that k is half-integer. Therefore, Z_{odd} only contains contributions from half-integer k. Picking $s = -2$,

$$\begin{aligned} Z_{\text{odd}}(\tau) &= q^{-2} \sum_{\substack{k \text{ half} \\ \text{integer}}} q^k \chi(\mathscr{M}_k) \\ &= q^{-2} \sum_{k \text{ half}} q^{\frac{2k-3}{2} + \frac{3}{2}} \chi(\mathrm{K3}^{[2k-3]}) \\ &= q^{-2} \sum_{n \text{ even}} q^{n/2+3/2} \chi(\mathrm{K3}^{[n]}) \\ &= q^{-1/2} \sum_{n \text{ even}} q^{n/2} \chi(\mathrm{K3}^{[n]}) \\ &= \frac{1}{2}\left(G(q^{1/2}) - G(-q^{1/2}) \right), \end{aligned} \tag{6.55}$$

where in the last equality we subtracted $G(-q^{1/2})$ to project onto factors with even n. Via a similar calculation one finds

$$\begin{aligned} Z_{\text{even}}(\tau) &= q^{-2} \sum_{k \text{ integer}} q^k \chi(\mathscr{M}_k) = q^{-1/2} \sum_{n \text{ odd}} q^{n/2} \chi(\mathrm{K3}^{[n]}) \\ &= \frac{1}{2}\left(G(q^{1/2}) + G(-q^{1/2})\right). \end{aligned} \tag{6.56}$$

Z_0 also contains terms with $w_2 = 0$ and even k, for which the description of the moduli space as a symmetric product does not work. For now we formally write

$$\begin{aligned} Z_0(\tau) &= F(q) + q^{-2} \sum_{k \text{ integer}} q^k \chi(\mathscr{M}_k) \\ &= F(q) + \frac{1}{2}\left(G(q^{1/2}) + G(-q^{1/2})\right), \end{aligned} \tag{6.57}$$

where the first term contains the corrections for the terms with even k. Collecting what we have found so far,

$$\begin{aligned} Z_{\mathrm{SU}(2)}(\tau) &= \frac{1}{2}F(q) + \frac{1}{4}\left(G(q^{1/2}) + G(-q^{1/2})\right), \\ Z_{\mathrm{SO}(3)}(\tau) &= F(q) + 2^{21}G(q^{1/2}) + 2^{10}G(-q^{1/2}). \end{aligned} \tag{6.58}$$

6.6.3 Determining Z_0

At this point, now that w and $F(q)$ are the only unknowns left to determine, we assume S-duality. We first consider the effect of the S-duality transformations on the three partition functions Z_0, Z_{even} and Z_{odd}. Pick any even v. From equation (6.27) it follows that

$$\begin{aligned} Z_{\text{even}}(-\tau^{-1}) &= Z_v(-\tau^{-1}) = 2^{-11}(-i\tau)^{w/2} \sum_u (-1)^{Q(u,v)} Z_u(\tau) \\ &= 2^{-11}(-i\tau)^{w/2} \left(Z_0(\tau) + \sum_{u \text{ even}} (-1)^{Q(u,v)} Z_{\text{even}} + \sum_{u \text{ odd}} (-1)^{Q(u,v)} Z_{\text{odd}} \right) \\ &= 2^{-11}(-i\tau)^{w/2} \left((-n_{\text{even}}^{Q(u,v)=1} + n_{\text{even}}^{Q(u,v)=0}) Z_{\text{even}} + (-n_{\text{odd}}^{Q(u,v)=1} + n_{\text{odd}}^{Q(u,v)=0}) Z_{\text{odd}} \right). \end{aligned}$$

Here $n_{\text{even}}^{Q(u,v)=k}$ is the number of even u's for which $Q(u,v) = k$, and similarly for $n_{\text{odd}}^{Q(u,v)=k}$. These numbers may be evaluated using considerations similar to those used to compute n_{even} and n_{odd}. Applying the same procedure to Z_{odd}, we find that the effect of S-duality in our case is

$$\begin{aligned} Z_0(-\tau^{-1}) &= 2^{-11}(-i\tau)^{w/2} Z_{\mathrm{SO}(3)}(\tau), \\ Z_{\text{even}}(-\tau^{-1}) &= 2^{-11}(-i\tau)^{w/2} \left(Z_0(\tau) + (2^{10}-1) Z_{\text{even}}(\tau) - 2^{10} Z_{\text{odd}}(\tau) \right), \\ Z_{\text{odd}}(-\tau^{-1}) &= 2^{-11}(-i\tau)^{w/2} \left(Z_0(\tau) + (-2^{10}-1) Z_{\text{even}}(\tau) + 2^{10} Z_{\text{odd}}(\tau) \right). \end{aligned} \tag{6.59}$$

By eliminating the unknown $F(q)$ from the above equations, we can determine w. Note that subtracting the lower two above equations give

$$Z_{\text{even}}(-\tau^{-1}) - Z_{\text{odd}}(-\tau^{-1}) = (-i\tau)^{w/2}(Z_{\text{even}}(\tau) - Z_{\text{odd}}(\tau)). \tag{6.60}$$

On the other hand we know that $Z_{\text{even}} - Z_{\text{odd}} = G(-q^{1/2})$, and via Proposition 6.17 we can calculate the effect of $-\tau^{-1}$ on this expression directly:

$$Z_{\text{even}}(\tau) - Z_{\text{odd}}(\tau) = G(-q^{1/2}) \to (-i\tau)^{-12} G(-q^{1/2})$$

$$= (-i\tau)^{-12}(Z_{\text{even}}(\tau) - Z_{\text{odd}}(\tau)), \tag{6.61}$$

from which it follows that $w = -24$. Putting this in the middle formula of (6.59) we find that S-duality demands the following

$$\begin{aligned} Z_{\text{even}}(-\tau^{-1}) &= 2^{-11}(-i\tau)^{-12}(F(q) + 2^{10}Z_{\text{even}} - 2^{10}Z_{\text{odd}}) \\ &= 2^{-11}(-i\tau)^{-12}\Big[F(q) + 2^9\left(G(q^{1/2}) + G(-q^{1/2})\right) \\ &\qquad\qquad - 2^9\left(G(q^{1/2}) - G(-q^{1/2})\right)\Big] \\ &= 2^{-11}(-i\tau)^{-12}\left(F(q) + 2^{10}G(-q^{1/2})\right). \end{aligned} \tag{6.62}$$

Now we can use Proposition 6.17 again in equation (6.56) to calculate explicitly how Z_{even} transforms:

$$Z_{\text{even}}(-\tau^{-1}) = \frac{1}{2}\left(2^{-12}(-i\tau)^{-12}G(q^2) + (-i\tau)^{-12}G(-q^{1/2})\right). \tag{6.63}$$

The last terms of the last two formulae already match, leaving us with the first two which finally give us

$$F(q) = \frac{1}{4}G(q^2). \tag{6.64}$$

Therefore, the partition functions of SU(2) and SO(3) theory on K3 surfaces are

$$\begin{aligned} Z_{\text{SU(2)}}(\tau) &= \frac{1}{8}G(q^2) + \frac{1}{4}\left(G(q^{1/2}) + G(-q^{1/2})\right), \\ Z_{\text{SO(3)}}(\tau) &= \frac{1}{4}G(q^2) + 2^{21}G(q^{1/2}) + 2^{10}G(-q^{1/2}). \end{aligned} \tag{6.65}$$

Finally, in so far as we have not assumed it, we should test whether everything is S-dual. We have derived Z_0 by assuming that Z_{even} is S-dual, which leaves Z_{odd} and Z_0 to check. As for the former, a calculation similar to (6.62) shows that it should transform according to

$$\begin{aligned} Z_{\text{odd}}(-\tau^{-1}) &= 2^{-11}(-i\tau)^{-12}\left(F(q) - 2^{10}G(-q^{-1/2})\right) \\ &= 2^{-11}(-i\tau)^{-12}\left(\frac{1}{4}G(q^2) - 2^{10}G(-q^{-1/2})\right). \end{aligned} \tag{6.66}$$

Using 6.17 on equation (6.55), we see that this is indeed the case:

$$Z_{\text{odd}}(-\tau^{-1}) = \frac{1}{2}\left(2^{-12}(-i\tau)^{-12}G(q^2) - (-i\tau)^{-12}G(-q^{1/2})\right). \tag{6.67}$$

Finally, we can write Z_0 as

$$Z_0(\tau) = \frac{1}{4}G(q^2) + Z_{\text{even}}(\tau), \tag{6.68}$$

which transforms as

$$\begin{aligned} Z_0(-\tau^{-1}) &= 2^{10}(-i\tau)^{-12}G(q^{1/2}) + 2^{-11}(-i\tau)^{-12}\left(\frac{1}{4}G(q^2) + 2^{10}G(-q^{1/2})\right) \\ &= 2^{-11}(-i\tau)^{-12}\left(\frac{1}{4}G(q^2) + 2^{21}G(q^{1/2}) + 2^{10}G(-q^{1/2})\right) \\ &= 2^{-11}(-i\tau)^{-12}Z_{\text{SO(3)}}(\tau). \end{aligned} \tag{6.69}$$

This is exactly what equation (6.59) predicts it to be.

Chapter 7

Conclusion

In this thesis, we have studied topologically twisted $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$ theories on K3 surfaces. We have seen that the partition function for $\mathrm{SO}(3)$ theory consists of 2^{22} terms, one for each element of $H^2(\mathrm{K3}, \mathbb{Z}_2)$. We were able to compute all but one of the 2^{22} terms of the $\mathrm{SO}(3)$ partition function; the last term, which also is the $\mathrm{SU}(2)$ partition function, was problematic because half of its terms involved moduli spaces which were not smooth. In order to address this problem and finish the calculations, we used S-duality, which determines the terms completely. All of the other terms indeed turned out to be S-dual, strongly suggesting that the Yang-Mills theory on K3 surfaces is indeed S-dual.

Vafa and Witten [1] studied the theory on several other spaces as well. In this thesis we have focused on the only Calabi-Yau two-folds which are both compact and simply connected. One can drop compactness and study so-called ALE spaces: minimal resolutions of orbifolds of the form $\mathbb{C}^2/\Gamma$, where Γ is a finite subgroup of $\mathrm{SU}(2)$. Testing S-duality on these spaces is not straightforward, however, because the modular properties (6.27) that the partition function must satisfy assume a compact base space. Later it was found that due to the structure of these spaces, S-duality does not hold in this case [44]. Another space which Vafa and Witten used was $\mathbb{CP}^2$, on which S-duality does seem to hold. Interestingly, however, in this case there is a holomorphic anomaly: the partition functions Z_v (with $v \in H^2(\mathbb{CP}^2, \mathbb{Z}_2)$) are not holomorphic but also have a $\bar{\tau}$-dependence. One can go further, however, and for prime N derive the partition function for general Kähler four-manifolds with $h^{2,0} \neq 0$. Then one can generalize this to gauge group $\mathrm{SU}(N)$ (see for example [45]).

There are a number of fascinating connections with other areas. In 1988, Witten twisted $\mathcal{N} = 2$ Yang-Mills theory, obtaining a topological theory which was independent of the coupling constant g [46]. By taking the limit $g \to 0$, he found an expression for Donaldson's invariants of smooth four-manifold in terms of expectation values of certain operators. Later on, by exploiting the independence of the theory on the coupling constant in this twisted theory, Seiberg and Witten [47] moved to the strongly coupled region, and constructed the so-called Seiberg-Witten invariants, which essentially contain the same information as the Donaldson invariants but are much easier to deal with.

In studying S-duality in section 6.4 and later, we have already briefly touched the Langlands program, through the group $\widehat{G}$, which is the Langlands dual of the gauge group G. There is another connection, however, that goes much further. Recall that we obtained the topologically twisted theory by twisting the right hand factor $\mathrm{SU}(2)_{\mathrm{R}}$ of the structure group with the R-symmetry group $\mathrm{SU}(4)$ that comes from the $\mathcal{N} = 4$ supersymmetry. In the case $\mathcal{N} = 4$, however, there are three inequivalent ways of doing such a twist. It turns out that if one applies one of the other twists to just the right problem, then the geometric Langlands program arises naturally [48]. Seemingly esoteric notions such as Hecke eigensheaves, $\mathcal{D}$-modules, and so on, appear spontaneously in the physics, with new insights about their properties.

In string theory there are also S-dualities. In particular, S-duality plays an important role in M-theory, because it connects type I string theory with $\mathrm{SO}(32)$ heterotic string theory, and type IIB string theory with itself. Each time it sends the coupling constant to its inverse. Furthermore, it also relates type IIA string theory and $E_8 \times E_8$ heterotic string theory to M-theory (see for example [49]).

There are also more direct links with string theory. Indeed, if one takes a minimal supersymmetric Yang-Mills theory in 10 dimensions, one obtains a theory that contains a gauge field A, which is a connection on a G-bundle, and a fermion field which is a positive chirality spinor field with values in $\mathrm{ad}\, G$. If one then applies a dimensional reduction to $d = 4$ by simply assuming all the fields to be independent of the dimensions $x^4, \ldots, x^9$, one reduces the structure group of the manifold from $\mathrm{SO}(10)$ to $\mathrm{SO}(4) \times \mathrm{SO}(6)$. Allowing for the presence of spinors the right factor becomes $\mathrm{Spin}(6) = \mathrm{SU}(4)$, which is just the R-symmetry group of $\mathcal{N} = 4$ supersymmetry. Thus by applying this dimensional reduction one obtains a $\mathcal{N} = 4$ supersymmetric Yang-Mills theory on a four-manifold. An example of the relevance of twisted theories in string theory comes from the fact that $\mathrm{U}(N)$ twisted theory is embedded as the topological subsector of $\mathcal{N} = 4$ super Yang-Mills theory on N D4-branes, and there exists a duality between twisted Yang-Mills theory and a conformal field theory of free fermions on the two-torus (see for example [50]). Furthermore, there are some hints for deeper connections with string theory. For example, the function $G(q) = \eta(\tau)^{-24}$, which played an important role in our calculations, is exactly the partition function of the left-moving modes of the bosonic string, and the fact that the cohomology of K3 surfaces is 24-dimensional matches the fact that in bosonic string theory, there are 24 physical states (namely $\alpha^i_{-1} |0\rangle$, where i runs over the transverse directions in the light-cone gauge).

Summarizing, we see that each of the main ingredients of this thesis (Yang-Mills theory, topological twisting, and S-duality) play central roles in various areas of both physics and mathematics, and bind them together by inspiring new intuitions and theories in both of them. The way they have combined in this thesis is just one of the possible ways, applied to a specific situation. There are likely plenty of other ideas left to pursue.

Appendix A

Complex geometry

Here we give a brief introduction to complex differential geometry. For more information, see [7, 12–14].

A.1 Complex manifolds

Definition A.1 A manifold is said to be *almost complex* if there exists a smooth bundle endomorphism $J : TM \to TM$ such that $J^2 = -1$. Such a manifold necessarily has even dimension and is orientable.

Definition A.2 A *complex manifold* is one whose atlas of charts are in $\mathbb{C}^n$ for some n, with holomorphic transition maps. Such a manifold is an ordinary manifold of dimension $2n$, and is almost complex.

Proposition A.3 *Let $G \times M \to M$ be the proper and free action of a complex Lie group G on a complex manifold M. Then the quotient M/G is a complex manifold in a natural way and the quotient map $\pi : M \to M/G$ is holomorphic.*

Let M be complex and n-dimensional and $z \in M$. Then there are three distinct notions of tangent spaces to M at z:

- The usual real tangent space T_zM when considering M as a $2n$-dimensional smooth manifold.
- The complexified tangent space, $T_zM^{\mathbb{C}} := T_zM \otimes \mathbb{C}$. If we have local coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$, then this space is spanned by $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\}_{i=1,\ldots,n}$ (with complex coefficients). Alternatively, we may take the frame consisting of

$$\frac{\partial}{\partial z^i} := \frac{1}{2}\left(\frac{\partial}{\partial x^i} - i\frac{\partial}{\partial y^i}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} + i\frac{\partial}{\partial y^i}\right). \tag{A.1}$$

- The *holomorphic tangent space* at z, denoted $T_{M,z}$, is the space $T_{M,z} := \mathrm{span}_{\mathbb{C}}\{\frac{\partial}{\partial z_i}\}_i$. It is the space of derivations that vanish on antiholomorphic functions (making it independent on the chosen coordinate system). Similarly, one can define the antiholomorphic tangent space $T'_{M,z} = \mathrm{span}_{\mathbb{C}}\{\frac{\partial}{\partial \bar{z}_i}\}_i$; clearly then $T_zM^{\mathbb{C}} = T_{M,z} \oplus T'_{M,z}$.

If we extend J_z linearly to $T_zM^{\mathbb{C}}$, then it has eigenvalues i and $-i$. Accordingly, $T_zM^{\mathbb{C}}$ decomposes into two eigenspaces. It is not hard to show that the eigenspace having eigenvalue i is the holomorphic tangent bundle $T_{M,z}$, and the eigenspace having eigenvalue $-i$ is $T'_{M,z}$.

The dual of $T_zM^{\mathbb{C}}$ is the complexified cotangent space, or in other words $(T_z^*M)^{\mathbb{C}} = (T_zM^{\mathbb{C}})^*$. We may take as a basis for it $\mathrm{d}z^i := \mathrm{d}x^i + i\mathrm{d}y^i$ and $\mathrm{d}\bar{z}^i := \mathrm{d}x^i - i\mathrm{d}y^i$, with $i \in \{1, \ldots, n\}$.

If a differential form $\omega \in \Omega^{p+q}(M)^{\mathbb{C}}$ may be written as

$$\omega = \sum_{i_k, j_l} \omega_{i_1,\dots,i_p,\bar{j}_1,\dots,\bar{j}_q} \mathrm{d}z^{i_1} \wedge \cdots \wedge \mathrm{d}z^{i_p} \wedge \mathrm{d}\bar{z}^{j_1} \wedge \cdots \wedge \mathrm{d}\bar{z}^{j_q}, \tag{A.2}$$

then ω is said to be of type (p,q) (note that the coefficient functions do not have to be (anti-)holomorphic). Defining $\Omega^{p,q}(M) := \{\omega \in \Omega^{p+q} \mid \omega \text{ is of type } (p,q)\}$, we have the decomposition

$$\Omega^r(M)^{\mathbb{C}} = \bigoplus_{p+q=r} \Omega^{p,q}(M). \tag{A.3}$$

Let M be a complex manifold. Defining the *Dolbeault operators* for a (p,q)-form,

$$\begin{aligned} \partial &:= \pi_{p+1,q} \circ \mathrm{d} : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M), \\ \bar{\partial} &:= \pi_{p,q+1} \circ \mathrm{d} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M), \end{aligned} \tag{A.4}$$

we have that

$$\begin{aligned} \mathrm{d} &= \partial + \bar{\partial}, \\ \partial^2 = \bar{\partial}^2 &= \partial\bar{\partial} + \bar{\partial}\partial = 0. \end{aligned} \tag{A.5}$$

Now let $X \subset M$ be a complex submanifold. Then there is a natural injection $T_X \subset T_M|_X$.

Definition A.4 The *normal bundle* N_X of X in M is the cokernel of the natural injection $T_X \subset T_M|_X$. Thus, by definition the normal bundle is such that there is a short exact sequence of vector bundles, called the *normal bundle sequence*

$$0 \to T_X \to T_M|_X \to N_X \to 0.$$

Theorem A.5 **Adjunction formula.** *Let $X \subset M$ be a complex submanifold of M. Then*

$$K_X = K_M|_X \otimes \det(N_X). \tag{A.6}$$

A.2 Hermitian and Kähler manifolds

Definition A.6 If g satisfies for all $x \in M$ and all $X, Y \in T_x M$

$$g(JX, JY) = g(X, Y), \tag{A.7}$$

then g is said to be *Hermitian*. In this case, M is called a *Hermitian manifold*.

Two holomorphic tangent vectors are orthogonal with respect to a Hermitian manifold, and similarly for two anti-holomorphic vectors. Every complex manifold admits a Hermitian metric by

$$h(X,Y) := \frac{1}{2}(g(X,Y) + g(JX,JY)).$$

We now define the *fundamental form* ω by

$$\omega(X,Y) := g(JX,Y). \tag{A.8}$$

In coordinates, this is $\omega = 2ig_{\mu\bar{\nu}}\, \mathrm{d}z^\nu \wedge \mathrm{d}\bar{z}^\nu$. ω is indeed antisymmetric: $\omega(X,Y) = g(JX,Y) = g(J^2X, JY) = -g(X,JY) = -g(JY,X) = -\omega(Y,X)$. Moreover, ω is non-degenerate, and it is compatible with J, in the sense that

$$\begin{aligned} \omega(JX,JY) &= g(J^2X,JY) = g(JX,Y) = \omega(X,Y), \\ \omega(X,JY) &= g(JX,JY) = g(X,Y) > 0. \end{aligned} \tag{A.9}$$

Definition A.7 Let M be a Hermitian manifold with fundamental form ω. If ω is closed,

$$\mathrm{d}\omega = 0, \tag{A.10}$$

then M is called a *Kähler manifold*, and its metric a *Kähler metric*.

Proposition A.8 *Let ∇ be the Levi-Civita connection associated to g on a Hermitian manifold M. The following are equivalent:*

1. *M is Kähler,*
2. *$\nabla J = 0$,*
3. *$\nabla \omega = 0$,*
4. *The holonomy group of ∇ is contained in $\mathrm{U}(n)$.*

Proposition A.9 *Let $g_{\mu\bar{\nu}}$ be the coefficients of the metric (these are the only coefficients, because $g_{\mu\nu} = g_{\overline{\mu\nu}} = 0$ because a pair of two (anti-)holomorphic vectors are orthogonal.) Then g locally satisfies*

$$g_{\mu\bar{\nu}} = \frac{\partial^2 K}{\partial z^\mu \partial \bar{z}^\nu} \tag{A.11}$$

for some function K. This function is called the Kähler potential.

Conversely, if a metric is locally given by $g_{\mu\bar{\nu}} = \partial_\nu \partial_{\bar{\nu}} K$, then ω is locally given by $\omega = i\partial\bar{\partial}K$, which is closed because $\partial\bar{\partial} = -\frac{1}{2}\mathrm{d}(\partial - \bar{\partial})$, i.e. ω is locally exact. If we have an open covering, and a set of Kähler potentials K_i on it satisfying $K_i(z,\bar{z}) = K_j(z,\bar{z}) + f_{ij}(z) + \bar{f}_{ij}(\bar{z})$, then they define a global fundamental form ω.

Since all of the structures defining a Kähler manifold are essentially local, we can just restrict g, J and ω to Y:

Proposition A.10 *If $Y \subset X$ is a complex submanifold of the Kähler manifold X, then Y is Kähler.*

Proposition A.11 *For a* d*-closed form ω of type (p,q) on a Kähler manifold M, the following are equivalent:*

1. *ω is* d*-exact,*
2. *ω is ∂-exact,*
3. *ω is $\bar{\partial}$-exact,*
4. *ω is $\partial\bar{\partial}$-exact.*

Definition A.12 A *Calabi-Yau manifold* of dimension n is a compact Kähler manifold of dimension n having a trivial canonical bundle.

Proposition A.13 *A compact n-dimensional Kähler manifold is Calabi-Yau if and only if one of the following equivalent conditions hold:*

- *The canonical bundle of M is trivial,*
- *The structure group of M can be reduced from $\mathrm{U}(n)$ to $\mathrm{SU}(n)$,*
- *The first integral Chern class $c_1(M) \in H^2(M,\mathbb{Z})$ of M vanishes.*
- *The holonomy of the Kähler metric of M is contained in $\mathrm{SU}(n)$.*

A.3 Invariants

Definition A.14 Let M be a differentiable manifold. The k-th *Betti number* of M is $b_k(M) := \dim_{\mathbb{R}} H^k(M,\mathbb{R})$, where $H^k(M,\mathbb{R})$ is the de Rham cohomology. The (topological) *Euler characteristic* of M is $\chi(M) := \sum_i (-1)^i b_i(M)$.

Consider the *Dolbeault complex*

$$\Omega^{p,0}(M) \xrightarrow{\bar{\partial}} \Omega^{p,1}(M) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \Omega^{p,n}(M). \tag{A.12}$$

Definition A.15 The *Dolbeault cohomology* is the cohomology of the Dolbeault complex,

$$H^{p,q}(M) := \frac{\ker(\bar{\partial} : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M))}{\bar{\partial}\Omega^{p,q-1}}. \tag{A.13}$$

The dimensions $h^{p,q} := \dim H^{p,q}(M)$ are called the *Hodge numbers*.

Theorem A.16 *Let M be a compact Kähler manifold. Then*

$$H^k(M, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(M). \tag{A.14}$$

This decomposition does not depend on the chosen Kähler structure. Furthermore, we have

$$\overline{H}^{p,q}(M) = H^{q,p}(M). \tag{A.15}$$

Corollary A.17 *If M is compact, Kähler, and has complex dimension n, then*

$$\begin{aligned} b_k(M) &= \sum_{p+q=k} h^{p,q}(M), & h^{1,0}(M) &= \tfrac{1}{2} b_1(M), \\ h^{p,q}(M) &= h^{q,p}(M), & h^{p,q}(M) &= h^{n-p,n-q}(M). \end{aligned} \tag{A.16}$$

Appendix B

The Hodge dual

Let M be a n-dimensional pseudo-Riemannian orientable manifold, and let g be its metric tensor which has signature (p,q). Then locally, there is a frame $\{e\}$ (i.e. a collection of smooth sections which form a basis of the tangent space T_xM above each $x \in M$) of the tangent spaces such that g is diagonalized with respect to $\{e\}$, i.e. $g(e_\alpha, e_\beta) = \operatorname{diag}(1,\ldots,1,-1,\ldots,-1)$. This basis defines a dual basis $\{\theta^\alpha\}$ of the one-forms by $\langle \theta^\alpha, e_\beta\rangle = \delta^\alpha_\beta$. The metric can then be expressed by

$$g = \sum_{i=\alpha}^{n} \epsilon_\alpha \theta^\alpha \otimes \theta^\alpha, \tag{B.1}$$

where ϵ_α are the signs of the metric, corresponding to its signature. In addition, the volume form takes the form

$$\mathrm{dVol} = \theta^1 \wedge \cdots \wedge \theta^n. \tag{B.2}$$

If I is a multi-index of length k, i.e. $I = \{i_1, \ldots, i_k\}$ such that $i_1 < \cdots < i_k$, let $\theta^I := \theta^{i_1} \wedge \cdots \wedge \theta^{i_k}$, and let $\epsilon(I) = \epsilon_{i_1} \cdots \epsilon_{i_k}$. These θ^I's form a basis for the space of k-forms. Then the metric g on M defines an inner product on the space of k-forms[1], by declaring the θ^I to be orthonormal with respect to it:

$$\langle \theta^I, \theta^J \rangle = \epsilon(I)\delta^{I,J}, \tag{B.3}$$

and extending bilinearly to arbitrary k-forms. It may be shown that this inner product agrees with itself on overlaps, so that it is well defined on all of $\Omega^k(M)$. In coordinates, if the metric is $g_{\mu\nu}\, \mathrm{d}x^\nu \otimes \mathrm{d}x^\mu$, then this inner product takes the form

$$\langle \alpha, \beta \rangle = \sum_{\nu_1 < \cdots < \nu_k} \alpha^{\nu_1 \cdots \nu_k} \beta_{\nu_1 \cdots \nu_k} := \sum_{\nu_1 < \cdots < \nu_k} g^{\mu_1 \nu_1} \ldots g^{\mu_k \nu_k} \alpha_{\mu_1 \cdots \mu_k} \beta_{\nu_1 \cdots \nu_k}, \tag{B.4}$$

where $g^{\mu\nu} = (g^{-1})_{\mu\nu}$ is the inverse of the metric.

Definition B.1 Let $\alpha, \beta \in \Omega^k(M)$. The *Hodge dual* is a map $\star : \Omega^k(M) \to \Omega^{n-k}(M)$ defined by

$$\alpha \wedge \star \beta = \langle \alpha, \beta \rangle \, \mathrm{dVol}. \tag{B.5}$$

Proposition B.2 *Let $|I| = k$ denote the length of the multi-index I, and let $\bar{I}$ be the unique multi-index of length $n - |I|$ such that $I \cup \bar{I} = \{1, \ldots, n\}$ as sets. In addition, let $\sigma(I)$ be the sign of the permutation of $(1, \ldots, n)$ given by concatenating I and $\bar{I}$. Then*

$$\star \theta^I = \epsilon(I)\sigma(I)\theta^{\bar{I}}. \tag{B.6}$$

[1] Strictly speaking, this is not actually a metric (a function $\Omega^k(M)^2 \to \mathbb{R}$), but a function $\Omega^k(M)^2 \to C^\infty(M)$.

Proof By definition, we have

$$\theta^I \wedge \star\theta^I = \langle \theta^I, \theta^I \rangle \,\mathrm{dVol} = \epsilon(I)\,\mathrm{dVol} = \epsilon(I)\,\theta^1 \wedge \cdots \wedge \theta^n.$$

Since θ^I contains k θ's, $\star\theta^I$ has to contain all the others. Moreover, apparently it should be a multiple of $\epsilon(I)$. That is, $\star\theta^I$ has to be of the form $a\epsilon(I)\theta^{i_{k+1}} \wedge \theta^{i_{k+2}} \wedge \cdots \wedge \theta^{i_n} = a\epsilon(I)\theta^{\bar{I}}$, for some real a. Putting this in formula (B.5) and comparing with the formula above gives

$$\theta^I \wedge \star\theta^I = a\epsilon(I)\theta^{i_1} \wedge \cdots \wedge \theta^{i_k} \wedge \theta^{i_{k+1}} \wedge \cdots \wedge \theta^{i_n} = \epsilon(I)\theta^1 \wedge \cdots \wedge \theta^n.$$

This is precisely the case if $a = \sigma(I)$. □

In coordinates, this works out to be

$$\begin{aligned}\star\alpha &= \sum_{\substack{\mu_1<\cdots<\mu_n\\ \nu_1<\cdots<\nu_n}} \sqrt{|g|}\alpha_{\mu_1\cdots\mu_k}\epsilon^{\mu_1\cdots\mu_k}{}_{\nu_{k+1}\cdots\nu_n}\mathrm{d}x^{\nu_{k+1}} \wedge \cdots \wedge \mathrm{d}x^{\nu_n} \\ &= \frac{1}{k!(n-k)!}\sqrt{|g|}\alpha_{\mu_1\cdots\mu_k}\epsilon^{\mu_1\cdots\mu_k}{}_{\nu_{k+1}\cdots\nu_n}\mathrm{d}x^{\nu_{k+1}} \wedge \cdots \wedge \mathrm{d}x^{\nu_n} \end{aligned} \tag{B.7}$$

where the indices are raised by the metric as before, and where $|g|$ is the absolute value of the determinant of the metric.

Corollary B.3 *Under the conformal transformation $\tilde{g} := \lambda^2 g$, the Hodge dual when applied to a k-form transforms to*

$$\tilde{\star}\,\alpha = \lambda^{n-2k}\star a. \tag{B.8}$$

In particular, if M has an even dimension then $\star$ is conformally invariant on forms of middle degree.

Corollary B.4 *On k-forms, the Hodge dual squares to either plus or minus the identity*

$$\star^2 = (-1)^{q+k(n-k)}, \tag{B.9}$$

where q is the number of negative signs in the metric.

Proof We have

$$\star^2\theta^I = \epsilon(I)\epsilon(\bar{I})\sigma(I)\sigma(\bar{I})\theta^I.$$

Then the statement follows from $\epsilon(I)\epsilon(\bar{I}) = \epsilon_{i_1}\ldots\epsilon_{i_k}\epsilon_{i_{k+1}}\ldots\epsilon_n = 1^p(-1)^q = (-1)^q$ and $\sigma(I)\sigma(\bar{I}) = (-1)^{|I||\bar{I}|}$. □

Definition B.5 $\mathrm{d}^\dagger$ is the adjoint of d,

$$\langle \mathrm{d}\alpha, \beta \rangle = \langle \alpha, \mathrm{d}^\dagger\beta \rangle. \tag{B.10}$$

Theorem B.6 *If M is compact and without boundary, and if ω is a k-form, then*

$$\mathrm{d}^\dagger\omega = (-1)^{q+nk+n+1}\star\mathrm{d}\star\omega. \tag{B.11}$$

Of course, one could take this as the definition for $\mathrm{d}^\dagger$ instead, allowing it to be defined on manifolds which are not compact, and for forms which have no compact support. Definition B.5 would then follow as a proposition when M is compact, or when α and β have compact support.

When M is a $2n$-dimensional manifold, $\star$ is a linear map from $\Omega^n(M)$ to itself, which squares to either $+1$ or -1, depending on whether n is even or odd and on the signature of the metric. Thus, its eigenvalues are $+1, -1$ or $+i, -i$ respectively, and $\Omega^n(M)$ decomposes in the two corresponding eigenspaces. When $\star^2 = +1$, we make the following definition.

Definition B.7 Let $\omega \in \Omega^n(M)$. Then we say it is *self-dual* (*anti-self-dual*) if $\star\omega = \pm\omega$. Furthermore, we denote the self-dual part of ω by ω_+ and the anti-self-dual part of ω by ω_-.

Note that for any middle-degree form, we have $\omega_\pm = \frac{1}{2}(\omega \pm \star\omega)$, and $\omega = \omega_+ + \omega_-$ and $\star\omega = \omega_+ - \omega_-$.

When $\star^2 = -1$, the eigenvectors are of the form $\omega = \omega_i + \omega_{-i}$, $\star\omega = i(\omega_i - \omega_{-i})$ and $\omega_i = \frac{1}{2}(\omega - i\star\omega)$, $\omega_{-i} = \frac{1}{2}(\omega + i\star\omega)$.

Proposition B.8 *The decomposition of middle-degree forms into self-dual and anti-self-dual forms is orthogonal with respect to the inner product,*

$$\langle \omega_+, \omega_- \rangle = 0, \quad \text{or} \quad \langle \omega_i, \omega_{-i} \rangle = 0 \tag{B.12}$$

when $\star^2 = 1$ *or* $\star^2 = -1$*, respectively.*

Proof On the one hand,

$$\langle \omega_+, \omega_- \rangle \,\mathrm{dVol} = \omega_+ \wedge \star\omega_- = -\omega_+ \wedge \omega_-,$$

while on the other hand

$$\langle \omega_+, \omega_- \rangle \,\mathrm{dVol} = \langle \omega_-, \omega_+ \rangle \,\mathrm{dVol} = \omega_- \wedge \star\omega_+ = \omega_- \wedge \omega_+ = \omega_+ \wedge \omega_-.$$

The proof for the case $\star^2 = -1$ is similar. □

Lie-algebra valued forms

We let the Hodge star operate on a $\mathfrak{g}$-valued form by just letting it operate on the form-component,

$$\star\omega = \star(T_a \otimes \omega^a) = T_a \otimes \star\omega^a. \tag{B.13}$$

This means that the space of forms of middle degree decomposes into self-dual and anti-self-dual eigenvectors of $\star$ exactly as before.

If $\mathfrak{g}$ is a simple Lie-subalgebra of $\mathfrak{gl}(n, \mathbb{C})$, then $-\operatorname{Tr}(XY)$ is an inner product on $\mathfrak{g}$ (and hence positive definite). (This assumption will not be a restriction in our case.) This means that

$$-\operatorname{Tr}(\alpha \wedge \star\beta) = -\alpha^a \wedge \star\beta^b \operatorname{Tr}(T_a T_b) = -\langle \alpha^a, \beta^b \rangle \operatorname{Tr}(T_a T_b) \,\mathrm{dVol}$$

is positive definite. This allows us to define the following inner product:

$$(\alpha, \beta) := -\int_M \operatorname{Tr}(\alpha \wedge \star\beta) = -\int_M \langle \alpha^a, \beta^b \rangle \operatorname{Tr}(T_a T_b) \,\mathrm{dVol}. \tag{B.14}$$

Lemma B.9 *If* $\alpha \in \Omega^1(M, \mathfrak{g})$*, then with respect to this inner product the adjoint of the map* $[\alpha, \cdot]$ *is*

$$[\alpha, \cdot]^* = (-1)^{q+nk+n+1} \star [\alpha, \star\,\cdot]. \tag{B.15}$$

Proof We have

$$([\alpha, \beta], \gamma) = ([T_a, T_b]\alpha^a \wedge \beta^b, T_c\gamma^c) = -f^d_{ab} \operatorname{Tr}(T_d T_c) \int_M \alpha^a \wedge \beta^b \wedge \star\gamma^c$$

and on the other hand

$$\begin{aligned}(\beta, \star[\alpha, \star\gamma]) &= (T_b\beta^b, [T_a, T_c]\star(\alpha^a \wedge \star\gamma^c)) \\ &= -f^d_{ac} \operatorname{Tr}(T_b T_d) \int_M \beta^b \wedge \star^2\alpha \wedge \star\gamma^c\end{aligned}$$

$$= -(-1)^{q+nk+n+r-1}(-1)^{r-1} f^d_{ac} \operatorname{Tr}(T_b T_d) \int_M \alpha^a \wedge \beta^b \wedge \star\gamma^c.$$

Now,

$$\begin{aligned} f^d_{ac} \operatorname{Tr}(T_b T_d) &= \operatorname{Tr}(T_b[T_a, T_c]) = \operatorname{Tr}(T_b T_a T_c) - \operatorname{Tr}(T_b T_c T_a) \\ &= \operatorname{Tr}(T_b T_a T_c) - \operatorname{Tr}(T_a T_b T_c) = \operatorname{Tr}([T_b, T_a] T_c) \\ &= -f^d_{ab} \operatorname{Tr}(T_d T_c), \end{aligned}$$

so

$$(\beta, \star[\alpha, \star\gamma]) = -(-1)^{q+nk+n+1} f^d_{ab} \operatorname{Tr}(T_d T_c) \int_M \alpha^a \wedge \beta^b \wedge \star\gamma^c.$$

Moving the power of -1 to the other side and comparing with $([\alpha, \beta], \gamma)$ then completes the proof. □

Theorem B.10 *The adjoint of $\mathcal{D}_A$ is $\mathcal{D}_A^\dagger = (-1)^{q+nk+n+1} \star \mathcal{D}_A \star$.*

Proof This follows immediately from the previous theorem and from theorem B.6. □

Two-dimensional manifolds

Suppose M is a Riemannian orientable two-dimensional manifold. Then it is always possible to choose coordinates x, y such that the metric takes the form

$$g = \lambda^2(x, y)(\mathrm{d}x^2 + \mathrm{d}y^2) = \lambda^2(z, \bar{z})\, \mathrm{d}z\, \mathrm{d}\bar{z}. \tag{B.16}$$

Then it is a complex manifold; i.e. the atlas consists of functions from charts to $\mathbb{C}$ which are holomorphic on overlaps. As shown, on the space of one-forms the Hodge dual has eigenvalues $\pm i$. Let $\alpha = \alpha_x\, \mathrm{d}x + \alpha_y\, \mathrm{d}y \in \Omega^1(M)$. Then the Hodge dual of α takes the form

$$\begin{aligned} \star\alpha &= \sqrt{g} g^{\mu\rho} \epsilon_{\rho\nu} \alpha_\mu\, \mathrm{d}x^\nu = \sqrt{g}\,[-(g^{xy}\alpha_x + g^{yy}\alpha_y)\, \mathrm{d}x + (g^{xx}\alpha_x + g^{yx}\alpha_y)\, \mathrm{d}y] \\ &= \lambda^2(-\lambda^{-2}\alpha_y\, \mathrm{d}x + \lambda^{-2}\alpha_x \mathrm{d}x) = -\alpha_y\, \mathrm{d}x + \alpha_x\, \mathrm{d}y. \end{aligned} \tag{B.17}$$

Thus $\alpha = \pm i \star \alpha$ if and only if $\alpha_x = \mp i \alpha_y$. In particular, $\mathrm{d}z = \mathrm{d}x + i\mathrm{d}y$ is anti-self-dual, and $\mathrm{d}\bar{z} = \mathrm{d}x - i\mathrm{d}y$ is self-dual. In a sense, then, anti-self-duality is a generalization of the study of holomorphic forms.

Appendix C

Generalities

C.1 Lie groups and Lie algebras

Here, we assume that G is a matrix Lie group, and we write $\mathfrak{g} = \mathrm{Lie}(G)$. Recall that the adjoint map is defined for each $g \in G$ by $\mathrm{Ad}(g)(h) = ghg^{-1}$. The derivative of $\mathrm{Ad}(g)$ at e then is a map from $\mathfrak{g}$ to $\mathfrak{g}$, which we also denote by $\mathrm{Ad}(g)$. When G is a matrix group, we have in fact $\mathrm{Ad}(g)X = gXg^{-1}$.

Ad induces a map from $\mathfrak{g}$ to $\mathrm{Lie}(\mathrm{GL}(\mathfrak{g}))$, which we denote here by ad.[1] We have $\mathrm{ad}(X)Y = [X, Y]$ and $e^{\mathrm{ad}(X)} = \mathrm{Ad}(e^X)$.

Proposition C.1 *If G is a connected Lie group, then the center of $\mathfrak{g}$ is the Lie algebra of $Z(G)$.*

Proof Recall that for matrix groups, $X \in \mathfrak{g}$ if and only if $e^{tX} \in G$ for all $t \in \mathbb{R}$. Thus

$$X \in \mathrm{Lie}(Z(G)) \iff e^{tX} \in Z(G) \iff ge^{tX} = e^{tX}g$$

for all $g \in G$ and $t \in \mathbb{R}$.

Let $X \in \mathrm{Lie}(Z(G))$, and let $s, t \in \mathbb{R}$ and $Y \in \mathfrak{g}$ be arbitrary; then $e^{tX}e^{sY} = e^{sY}e^{tX}$. Since G is connected this is equivalent with $[X, Y] = 0$; thus X is in the center of $\mathfrak{g}$.

Conversely, let X be in the center of $\mathfrak{g}$, i.e. $\mathrm{ad}(Y)X = 0$ for all Y. This implies $\mathrm{Ad}(e^Y)X = X$ for all Y, and since any $g \in G$ can be written as $g = e^{Y_1} \cdots e^{Y_n}$ because G is connected, we have in fact $\mathrm{Ad}(g)X = X$ for any $g \in G$. This easily generalizes to $\mathrm{Ad}(g)(tX) = tX$ for any $t \in \mathbb{R}$, and applying the exponential to both sides then gives

$$e^{tX} = e^{\mathrm{Ad}(g)(tX)} = ge^{tX}g^{-1}$$

so that $X \in \mathrm{Lie}(Z(G))$. $\square$

Proposition C.2 *Let G be compact. If $\mathfrak{a}$ is an abelian ideal of $\mathfrak{g}$, then it is contained in the center of $\mathfrak{g}$.*

Without proof, but see for example [51, p. 151].

Here, we call a Lie group semisimple if its Lie algebra is semisimple.

Proposition C.3 *Let G be a compact connected Lie group. Then G is semisimple if and only if the center of G is finite.*

Proof One direction is easy (and does not need the previous proposition): if G is semisimple then $\mathfrak{g}$ must be zero, which by by Proposition C.1 is equivalent with $Z(G)$ being finite. As to the other direction, if $Z(G)$ is finite then the center of $\mathfrak{g}$ is trivial. By Proposition C.2, any abelian ideal is therefore trivial, so $\mathfrak{g}$ is semisimple, so G is too. $\square$

---

[1] Note that this notation differs from that of the rest of the document, where $\mathrm{Ad}(g)h = ghg^{-1}$ for $h \in G$, and $\mathrm{ad}(g) : \mathfrak{g} \to \mathfrak{g}$ is the derivative of this map at e.

C.2 Isomorphisms of SU(2)

SU(2) consists of the 2×2 unitary matrices of unit determinant,

$$U \in \mathrm{SU}(2) \quad \Longleftrightarrow \quad U^\dagger = U^{-1},\ \det U = 1. \tag{C.1}$$

SU(2) is isomorphic (in various ways) to a number of other spaces, and which description of the group is the most appropriate depends on the situation. Here, we will outline some of them.

First, let σ_i with $i = 1, \ldots, 3$ be the Pauli matrices. These are simultaneously unitary and Hermitian (and therefore square to themselves). If we write $\sigma_\mu = (\sigma_i, I)$, with $\mu = 1, \ldots, 4$, then it is easy to see that $\{\sigma_\mu\}_\mu$ forms a complete basis of the space of complex 2×2 matrices. Write $U = U_\mu \sigma^\mu$, then demanding that U be unitary gives

$$\begin{aligned} I = U^\dagger U &= U_\mu U_\nu^* \sigma^\mu \sigma^\nu \\ &= U_i U_j^* \sigma^i \sigma^j + U_i U_4^* \sigma^i \sigma^4 + U_4 U_j^* \sigma^4 \sigma^j + U_4 U_4^* \sigma^4 \sigma^4 \\ &= (i U_i U_j^* \epsilon^{ij}{}_k + U_k U_4^* + U_4 U_k^*) \sigma^k + U_\mu (U^\mu)^* I, \end{aligned}$$

so $\sum_\mu |U_\mu|^2 = 1$ and the coefficient $i U_i U_j^* \epsilon^{ij}{}_k + U_k U_4^* + U_4 U_k^*$ has to vanish. If we now also demand that $\det U = 1$, then

$$1 = \det U = U_i^2 \det \sigma^i + U_4^2 \det I = U_4^2 - \sum_i U_i^2.$$

Combining $\sum_\mu |U_\mu|^2 = 1$ and $U_4^2 - \sum_i U_i^2 = 1$ gives $U_i \in i\mathbb{R}$ and U_4 in $\mathbb{R}$. The demand that the coefficient $i U_i U_j^* \epsilon^{ij}{}_k + U_k U_4^* + U_4 U_k^*$ be zero is then also satisfied, because the first term may be written as a sum of the imaginary parts of products of the form $U_i U_j^*$, which is zero, while the second term is the real part of $U_k U_4^*$ which is also zero. Thus any unitary matrix may be written as

$$U = i U_j \sigma^j + U_4 I \quad \text{with} \quad U_\mu \in \mathbb{R} \quad \text{and} \quad U_\mu U^\mu = 1. \tag{C.2}$$

Therefore, SU(2) is homeomorphic with the 3-sphere S^3. Since, moreover, $(i\sigma_j)^2 = -I$ for any j and $(i\sigma_1)(i\sigma_2)(i\sigma_3) = -(i\sigma_3)^2 = -I$, SU(2) is also isomorphic with the quaternions as topological groups.

Another approach is through the Lie algebra of SU(2). Defining

$$\tau_j = \frac{\sigma_j}{2i}, \tag{C.3}$$

we see that τ_j is antihermitian and traceless, and

$$[\tau_i, \tau_j] = \epsilon_{ij}{}^k \tau_k, \tag{C.4}$$

so that $\{\tau_i\}_i$ spans $\mathfrak{su}(2)$. Since SU(2) is compact and connected, any $U \in \mathrm{SU}(2)$ may therefore be written as

$$U = e^u \tag{C.5}$$

for some $u \in \mathfrak{su}(2)$. (For compact connected Lie groups, the exponential map is surjective.)

C.3 The Pfaffian

Let A be a real skew-symmetric matrix, i.e. $A^\top = -A$. Then A has purely imaginary eigenvalues: if x is a complex eigenvector of norm 1 having eigenvalue λ, then

$$\overline{\lambda} = x^\dagger \overline{\lambda} x = (x^\dagger \lambda x)^\dagger = (x^\dagger A x)^\dagger = x^\dagger A^\dagger x = x^\dagger(-A)x = -\lambda.$$

Moreover, if x is an eigenvector having eigenvalue $i\lambda$ (now with $\lambda \in \mathbb{R}$, then x^* is an eigenvector having eigenvalue $-i\lambda$. Thus the eigenvalues come in pairs. When n is odd, one of these is paired with 0. Therefore, in this case, $\det A = 0$.

Since the eigenvalues are complex, it is not possible to diagonalize A to a real matrix. However, it is possible via an orthogonal matrix to bring it in the following block-diagonal form:

$$A' = \begin{pmatrix} 0 & \lambda_1 & & & & & & \\ -\lambda_1 & 0 & & & & & & \\ & & \ddots & & & & & \\ & & & 0 & \lambda_k & & & \\ & & & -\lambda_k & 0 & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{pmatrix}$$

with the λ_i as before.

Definition C.4 The *Pfaffian* of an $n \times n$ skew-symmetric matrix is, when writing it in the form above, defined as

$$\operatorname{Pf} A = \prod_{i=1}^{n} \lambda_k. \tag{C.6}$$

From the definition it is immediately clear that $(\operatorname{Pf} A)^2 = \det A$. Moreover, $\operatorname{Pf}(A^\top) = (-1)^n \operatorname{Pf} A$, $\operatorname{Pf}(\lambda A) = \lambda^n \operatorname{Pf} A$ and $\operatorname{Pf}(BAB^\top) = \det(B)\operatorname{Pf}(A)$ for arbitrary λ en B. From the last formula it follows that the Pfaffian is not completely coordinate-independent, but it does not change under basis transformations which have unit determinant.

Of course, there are alternative ways to calculate the Pfaffian. Let A be $2n \times 2n$-dimensional. Consider the subset Π of S_{2n} consisting of elements of the form

$$\begin{pmatrix} 1 & 2 & \dots & 2n-1 & 2n \\ i_1 & j_1 & \dots & i_n & j_n \end{pmatrix}$$

with $i_k < j_k$ and $i_1 < \dots < i_n$. Then we have

$$\begin{aligned} \operatorname{Pf}(A) &= \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \operatorname{sgn} \sigma \prod_{i=1}^{n} A_{\sigma(2i-1),\sigma(2i)} \\ &= \sum_{\sigma \in \Pi} \operatorname{sgn} \sigma \prod_{i=1}^{n} A_{\sigma(2i-1),\sigma(2i)} \end{aligned} \tag{C.7}$$

Also, let A be a skew-symmetric matrix acting on some $2n$-dimensional vector space V. Let $\{\psi\}$ be a basis for V. Define $\omega := \sum_{i<j} A_{ij} \psi^i \wedge \psi^j$. Then

$$\frac{1}{n!}\omega^n = \operatorname{Pf}(A)\, \psi_1 \wedge \dots \wedge \psi_{2n}. \tag{C.8}$$

Writing $\omega = \frac{1}{2}\psi^\top A\psi$ (in the sense of matrix multiplication and the wedge product), this expression can also be written in terms of fermionic integration, as follows:

$$\int \mathrm{d}\psi\, e^{\omega} = \int \mathrm{d}\psi\, e^{\frac{1}{2}\psi^\top A\psi} = \mathrm{Pf}\, A, \tag{C.9}$$

which we recognize as a variant on the commuting case and symmetric A

$$\int e^{-\frac{1}{2}x^\top Ax}\mathrm{d}^n x = \sqrt{\frac{(2\pi)^n}{\det A}}. \tag{C.10}$$

C.4 Čech cohomology with $\mathbb{Z}_2$ coefficients

The Čech cohomology is a cohomology based on the intersection properties of open covers of a topological space, taking coefficients in abelian groups. In the case of smooth manifolds it is naturally isomorphic to the singular cohomology, and when the coefficients are $\mathbb{R}$ then it coincides with the De Rham cohomology. We only need to deal with Čech cohomology with coefficients in $\mathbb{Z}_2$, which is particularly easy to describe.

Let $\mathbb{Z}_2$ be the group consisting of two elements, written multiplicatively, i.e. $\mathbb{Z}_2 = \{-1, 1\}$. Let $\mathcal{U} = \{U_i\}_i$ be an open cover of M. An r-simplex of this open cover is an ordered collection of $r + 1$ sets from the open cover such that their intersection is non-empty.

Definition C.5 Let $\sigma = (U_{i_0}, \ldots, U_{i_r})$ be such an r-simplex. Then an *r-cochain* is an element $f(i_0, \ldots, i_r) \in \mathbb{Z}_2$ which is totally symmetric under permutations on its arguments. The multiplicative group of Čech r-cochains is denoted by $\check{C}^r(\mathcal{U}, \mathbb{Z}_2)$.

Definition C.6 We define the coboundary operator δ on r-cochains by

$$(\delta f)(i_0, \ldots, i_{r+1}) = \prod_{j=0}^{r+1} f(i_0, \ldots, \widehat{i_j}, \ldots, i_{r+1}), \tag{C.11}$$

where the hat indicates that the parameter below it is omitted.

Proposition C.7 *The coboundary operator squares to the identity operator.*

Proof Consider a factor $f(i_0, \ldots, \widehat{i_j}, \ldots, \widehat{i_k}, \ldots, i_{r+2})$. This factor can only be 1 or -1. Suppose it is -1. Then the symmetry of f implies that there is also a factor $f(i_0, \ldots, \widehat{i_k}, \ldots, \widehat{i_j}, \ldots, i_{r+2}) = -1$, so that the two factors -1 cancel. Therefore, all factors -1 cancel against each other. □

Definition C.8 The *Čech cohomology* of $\mathcal{U}$ is the cohomology $\check{H}^*(\mathcal{U}, \mathbb{Z}_2)$ defined by $\check{C}(\mathcal{U}, \mathbb{Z}_2)$ and δ.

Now let $\mathcal{V}$ be a refinement of $\mathcal{U}$; then there is an induced map $\check{H}^p(\mathcal{U}, \mathbb{Z}_2) \to \check{H}^p(\mathcal{V}, \mathbb{Z}_2)$, leading to a direct system of abelian groups. The Čech cohomology of M is then defined by

$$\check{H}^p(M, \mathbb{Z}_2) = \varinjlim_{\mathcal{U}} \check{H}^p(\mathcal{U}, \mathbb{Z}_2). \tag{C.12}$$

We mention here that the Čech cohomology group can be defined for any abelian group G, and over topological spaces.

Theorem C.9 *If M is a smooth manifold then $\check{H}^*(M, \mathbb{R}) \approx H_{dR}(M, \mathbb{R})$.*

Here we also state a specific version of the universal coefficient theorem for simplicial (co)homology.

Theorem C.10 *If $H_i(M, \mathbb{Z})$ has no torsion, for example when M is simply connected, we have*

$$H_i(M, \mathbb{Z}) \otimes_{\mathbb{Z}} G \approx H_i(M, G), \tag{C.13}$$

although this isomorphism is not natural. By Poincaré duality a similar statement holds in cohomology.

Index

Bibliography

[1] C. Vafa and E. Witten, "*A strong coupling test of S-duality*," Nuclear Physics B **431** (1994) no. 1-2, 3–77, arXiv:hep-th/9408074.

[2] M. Atiyah, R. Dijkgraaf, and N. Hitchin, "*Geometry and Physics*," Phil. Trans. R. Soc. A **368** (2010) no. 1914, 913–926.

[3] N. Steenrod, *The Topology of Fibre Bundles*. Princeton University Press, 1999.

[4] A. Collinucci and A. Wijns, "*Topology of Fibre bundles and Global Aspects of Gauge Theories*," arXiv:hep-th/0611201v1.

[5] M. Nakahara, *Geometry, Topology and Physics*. Taylor & Francis Group, 2 ed., 2003.

[6] J. M. Lee, *Introduction to Smooth Manifolds*. Springer, 2002.

[7] A. Scorpan, *The Wild World of 4-Manifolds*. American Mathematical Society, 2005.

[8] Y. Choquet-Bruhat and C. Dewitt-Morette, *Analysis, Manifolds and Physics*. North Holland, 2 ed., 2004.

[9] S. K. Donaldson and P. B. Kronheimer, *The Geometry of Four-Manifolds*. Oxford University Press, 1990.

[10] R. E. Gompf and A. Stipsicz, *4-manifolds and Kirby calculus*. American Mathematical Society, 1999.

[11] A. Dold and H. Whitney, "*Classification of oriented sphere bundles over a 4-complex*," Annals of Mathematics **69** (1959) no. 3, 667–677.

[12] D. Huybrechts, *Complex Geometry: An Introduction*. Springer, 2004.

[13] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*. Wiley-Interscience, 1978.

[14] W. Barth, K. Hulek, C. Peters, and A. van de Ven, *Compact complex surfaces*. Springer, 2 ed., 2004.

[15] Y.-T. Siu, "*Every K3 surface is Kähler*," Inventiones Mathematicae **73** (1983) 139–150.

[16] A. Grothendieck, "*Sur quelques points d'algébre homologique*," Tokohu Math. J. **2** (1957) 119–221.

[17] T. Matumoto, *On Diffeomorphisms of a K3 Surface*. Kinokuniya, Tokyo, 1985.

[18] C. Borcea, "*Diffeomorphisms of a K3 surface*," Mathematische Annalen **275** (1986) 1–4. doi:10.1007/BF01458579.

[19] S. K. Donaldson, "*Polynomial invariants for smooth four-manifolds*," Topology **29** (1990) no. 3, 257–315.

[20] S. Vandoren and P. van Nieuwenhuizen, "*Lectures on instantons*," arXiv:0802.1862 [hep-th].

[21] M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld, and Y. I. Manin, "*Construction of instantons*," Physics Letters A **65** (1978) no. 3, 185 – 187.

[22] P. M. N. Feehan, *Geometry of the Moduli Space of Self-Dual Connections over the Four-Sphere*. PhD thesis, Columbia University, 1992.

[23] G. 't Hooft, "*Computation of the quantum effects due to a four-dimensional pseudoparticle*," Phys. Rev. **14D** (1976) no. 12, 3432–3450.

[24] A. A. Belavin, A. M. Polyakov, A. S. Schwartz, and Y. S. Tyupkin, "*Pseudoparticle solutions of the Yang-Mills equations*," Phys. Lett. **59B** (1975) no. 1, 85–87.

[25] D. S. Freed and K. K. Uhlenbeck, *Instantons and Four-Manifolds*. Springer-Verlag, 2 ed., 1990.
[26] H. B. Lawson, *The Theory of Gauge Fields in Four Dimensions*. American Mathematical Society, 1985.
[27] M. F. Atiyah, N. J. Hitchin, and I. M. Singer, "*Self-Duality in Four-Dimensional Riemannian Geometry*," Royal Society of London Proceedings Series A **362** (1978) 425–461.
[28] M. F. Atiyah and I. M. Singer, "*The Index of Elliptic Operators: III*," The Annals of Mathematics **87** (1968) no. 3, 546–604.
[29] S. Coleman, *Aspects of Symmetry*. Cambride University Press, 1985. Chapter 7, "The uses of instantons".
[30] A. Collinucci, *Instantons and cosmologies in string theory*. PhD thesis, University of Groningen, 2005. Chapter 2, "Instantons".
[31] A. Smilga, *Lectures on Quantum Chromodynamics*. World Scientific Publishing Co. Pte. Ltd., 2001.
[32] R. Jackiw and C. Rebbi, "*Vacuum Periodicity in a Yang-Mills Quantum Theory*," Phys. Rev. Lett. **37** (1976) no. 3, 172–175.
[33] C. G. Callan, R. F. Dashen, and D. J. Gross, "*The structure of the gauge theory vacuum*," Physics Letters B **63** (1976) no. 3, 334 – 340.
[34] M. Blau, "*The Mathai-Quillen Formalism and Topological Field Theory*," `arXiv:hep-th/9203026`.
[35] C. Montonen and D. Olive, "*Magnetic monopoles as gauge particles?*," Physics Letters B **72** (1977) no. 1, 117–120.
[36] S. Mukai, "*Symplectic structure of the moduli space of sheaves on an abelian or K3 surface*," Inventiones Mathematicae **77** (1984) 101–116.
[37] T. Nakashima, "*Moduli of stable rank two bundles with ample c_1 on K3 surfaces*," Archiv der Mathematik **61** (1993) 100–104.
[38] Z. Qin, "*Moduli of simple rank-2 sheaves on K3-surfaces*," Manuscripta Mathematica **79** (1993) 253–265.
[39] R. Dijkgraaf, "*Fields, Strings, Matrices and Symmetric Products*," `arXiv:hep-th/9912104`.
[40] F. Hirzebruch and T. Höfer, "*On the Euler number of an orbifold*," Mathematische Annalen **286** (1990) 255–260.
[41] I. G. Macdonald, "*The Poincaré Polynomial of a Symmetric Product*," Mathematical Proceedings of the Cambridge Philosophical Society **58** (1962) no. 04, 563–568.
[42] C. L. Siegel, "*A simple proof of $\eta(-1/\tau) = \sqrt{\tau/i}\eta(\tau)$*," Mathematika **1** (1954) 4.
[43] C. T. C. Wall, "*On the orthogonal groups of unimodular quadratic forms*," Mathematische Annalen **147** (1962) 328–338.
[44] M. Bianchi, F. Fucito, G. Rossi, and M. Martellini, "*Explicit construction of Yang-Mills instantons on ALE spaces*," Nuclear Physics B **473** (1996) no. 1-2, 367–404, `arXiv:hep-th/9601162v1`.
[45] J. M. F. Labastida and C. Lozano, "*The Vafa-Witten Theory for Gauge Group* $\mathrm{SU}(N)$," Adv. Theor. Math. Phys. **3** (1999) 1201, `arXiv:hep-th/9903172v1`.
[46] E. Witten, "*Topological quantum field theory*," Comm. Math. Phys. **117** (1988) no. 3, 353–386.
[47] N. Seiberg and E. Witten, "*Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD*," Nuclear Physics B **431** (1994) no. 3, 484–550.
[48] A. Kapustin and E. Witten, "*Electric-Magnetic Duality And The Geometric Langlands Program*," `hep-th/0604151`.
[49] K. Becker, M. Becker, and J. H. Schwarz, *String Theory and M-Theory: A Modern Introduction*. Cambridge University Press, 2007.
[50] L. Hollands, *Topological strings and Quantum Curves*. PhD thesis, University of Amsterdam, 2009.
[51] D. Bump, *Lie Groups*. Springer, 2004.